Proceedings of the International Congress of Mathematicians
August 16-24, 1983, Warszawa

HUZIHIRO ARAKI

The Work of Alain Connes

Theory of operator algebras, after being quietly nourished in somewhat isolation for 30 years or so, started a revolutionary development around late 1960's. Alain Connes came into this field just when the smokes of the first stage of the revolution were settling down. He immediately led the field to breathtaking achievements beyond the expectation of experts.

His most remarkable contributions are: (1) general classification and a structure theorem for factors of type III, obtained in his thesis [12], (2) classification of automorphisms of the hyperfinite factor [29], which served as a preparation for the next contribution, (3) classification of injective factors [31], and (4) applications of the theory of C^*-algebras to foliations and differential geometry in general [44, 50] — a subject currently attracting a lot of attention.

In this report, I shall mostly concentrate on the first three aspects which form a well-established and most spectacular part of the theory of von Neumann algebras.

1. Classification of type III factors

In the latter half of 1930's, Murray and von Neumann initiated the study of what are now called von Neumann algebras (i.e. weakly closed *-subalgebra of the *-algebra $L(H)$ of all bounded linear operators on a Hilbert space H) and classified the factors (i.e. von Neumann algebras with trivial centers) into the types I_n, $n = 1, 2, \ldots$, and I_∞ (isomorphic to $L(H)$ with $\dim H = n$ and ∞), II_1, II_∞ and III. (In the following we restrict our attention to von Neumann algebras M on separable Hilbert spaces H.) Only three type III factors (and only three type II_1 factors) had been known to be mutually non-isomorphic till 1967, when Powers showed the existence of a continuous family of mutually non-isomorphic type III factors.

Traces provided a tool for a systematic analysis of type II factors at an earlier stage, while the non-existence of traces made type III factors remain untractable till late 1960's, when the Tomita–Takesaki theory was created and furnished a powerful tool for type III. To introduce notation, let M be a von Neumann algebra on a separable Hilbert space H and let $\Psi \in H$ be cyclic (i.e. $M\Psi$ be dense in H) and separating (i.e. such that $x\Psi = 0$ for $x \in M$ implies $x = 0$). The conjugate linear operator $S_\Psi x\Psi \equiv x^*\Psi$, $x \in M$, has a closure $\bar{S}_\Psi$ and defines the positive selfadjoint operator $\Delta_\Psi \equiv \bar{S}_\Psi^* \bar{S}_\Psi$, called the *modular operator*. The Tomita–Takesaki theory says that $x \in M$ implies $\sigma_t^\psi(x) \equiv \Delta_\Psi^{it} x \Delta_\Psi^{-it} \in M$. The one-parameter group of (*-)automorphisms σ_t^ψ of M depends only on the positive linear functional $\psi(x) \equiv (x\Psi, \Psi)$ and is called the *group of modular automorphisms.*

Connes [12] has shown that the modular automorphisms for different ψ's are mutually related by inner automorphisms (in other words, the independence of σ_t^ψ from ψ in the quotient Out M of the group Aut M of all automorphisms modulo the subgroup Int M of all inner automorphisms) and introduced the following two isomorphism invariants for M:

$$S(M) = \bigcap_\Psi \mathrm{Sp}\,\Delta_\Psi \quad \text{(Sp denotes the spectrum)},$$

$$T(M) = \{t \in \boldsymbol{R}:\ \sigma_t^\psi \in \mathrm{Int} M\}.$$

(In a more general case, $S(M) = \bigcap \mathrm{Sp}\,\Delta_\psi$, where the intersection is taken over faithful normal semifinite weights ψ.) It turns out that $S(M)\setminus\{0\}$ is a closed multiplicative subgroup of $\boldsymbol{R}_+^*$, and this leads to the classification of type III factors into the types III_λ, $0 \leqslant \lambda \leqslant 1$:

$$S(M) = \{\lambda^n:\ n \in \boldsymbol{Z}\} \cup \{0\} \quad \text{if } 0 < \lambda < 1,$$

$$S(M) = \boldsymbol{R}_+^* \quad \text{if } \lambda = 1, \qquad S(M) = \{0, 1\} \quad \text{if } \lambda = 0.$$

The Powers examples R_λ, $0 < \lambda < 1$, due to Powers, are of types III_λ and hence mutually non-isomorphic. The two invariants $r_\infty(M)$ and $\varrho(M)$ of Araki and Woods, introduced for a systematic classification of infinite tensor products of type I factors (including R_λ), are shown to be equivalent to $S(M)$ and $T(M)$ for them [7].

2. Structure analysis of type III factors

Connes [12] went on and succeeded in analysis of the structure of type III_λ factors M, $0 \leqslant \lambda < 1$, in terms of a type II von Neumann algebra N (with a non-trivial center) and an automorphism θ of N, such that M is the so-called *crossed product* $N \times_\theta \mathbf{Z}$ of N by θ.

For $0 < \lambda < 1$, θ should scale a trace τ of a type II_∞ factor N in the sense that $\tau(\theta x) = \lambda\tau(x)$ for all $x \in N_+$ and $N_i \times_{\theta_i} \mathbf{Z}$, $i = 1, 2$, are isomorphic if and only if there exists an isomorphism π of N_1 onto N_2 such that $\pi^{-1}\theta_2\pi\theta_1^{-1}$ is inner, or equivalently (in view of a later result of Connes and Takesaki), $\pi\theta_1\pi^{-1} = \theta_2$. This means that the pair (N, θ) is uniquely determined by M and the classification of M is reduced to that of the pair (N, θ).

For $\lambda = 0$, θ should scale down a trace τ of a type II_∞ von Neumann algebra N in the sense that $\tau(\theta x) \leqslant \varrho\tau(x)$ for all $x \in N_+$ for some $\varrho < 1$ and, again, there is a somewhat more complicated uniqueness result for the pair (N, θ).

Motivated by the above results of Connes, a general structure theorem including the type III_1 has been obtained by Takesaki in terms of a one-parameter group θ_t of trace-scaling automorphisms of a type II_∞ von Neumann algebra N.

In the process of developing the above classification and structure theory, Connes introduced two important technical tools, namely the unitary Radon–Nikodym cocycle (equivalently, relative modular operators) useful in application to quantum statistical mechanics, non-commutative L_p theory, etc., and the Connes spectrum useful in the analysis of C^* dynamical systems.

3. Classification of automorphisms of the hyperfinite factor

A von Neumann algebra, containing an ascending sequence of finite-dimensional subalgebras with a dense union, is called *approximately finite-dimensional* (AFD). AFD factors of type II_1, as shown by Murray and von Neumann, are all isomorphic to what is called the *hyperfinite factor*, denoted by R in the following. Connes [29] has given a complete classification of automorphisms of R modulo inner automorphisms (i.e. the conjugacy class of $\mathrm{Out}\, R$). Namely, a complete set of isomorphism invariants in $\mathrm{Out}\, R$ for an $a \in \mathrm{Aut}\, R$ is given by the pair of the outer period p $(= 2, 3, \ldots)$, which is the smallest $p > 0$ such that a^p is inner, defined to be 0 for outer aperiodic a, and the obstruction γ which is the

p-th root of 1 (1 for $p = 0$) such that $\alpha^p = \mathrm{Ad}\, U$, $\alpha(U) = \gamma U$, where $(\mathrm{Ad}\, U)(x) = UxU^*$. Although the result is about a specific R, this factor R is in the bottom of all known AFD factors and the result that outer aperiodic automorphisms of R are all conjugate up to inner automorphisms is essential for the results described in the next section.

As a by-product, Connes [23] solved negatively one of old problems on von Neumann algebras by exhibiting, for each $0 < \lambda < 1$, factors of type III_λ, not anti-isomorphic to themselves.

4. Classification of AFD factors

A complete classification of AFD factors of type III_λ, $\lambda \neq 1$, is what I consider the most distinguished work of Connes. It turns out that an AFD factor of type III_λ is unique and is isomorphic to R_λ for each $0 < \lambda < 1$, while AFD factors of type III_0 are isomorphic to Krieger's factors associated with single non-singular ergodic transformations of the Lebesgue measure space, their isomorphism classes being in one-to-one correspondence with the metric equivalence classes of non-singular non-transitive ergodic flows on the Lebesgue measure space.

One of the most important technical ingredients of the proof is the equivalence of various concepts about a von Neumann algebra which arose over years in theory of von Neumann algebras. Murray and von Neumann found a factor N of type II_1 non-isomorphic to R (distinguished by Property Γ). In 1962, Schwartz distinguished N, R and $N \otimes R$ by the following Property P: A von Neumann algebra M on a Hilbert space H has the property P iff for any $T \in L(H)$, the norm closed convex hull of the uTu^* with u varying over all unitary operators in M intersects M'. Any AFD factors posseses Property P.

The Property P for M implies the existence of a projection of norm 1 from $L(H)$ to M'. This is the Hakeda–Tomiyama extension property for M', called Property E by Connes, and is stable under taking the intersection of a decreasing family, the weak closure of the union of an ascending family, the commutant, tensor products and crossed product by an amenable group. Thus the Property P implies Property E for M.

Any projection E of norm 1 from a C^*-algebra $\mathfrak{A}_1$ to its subalgebra $\mathfrak{A}_2$ is shown by Tomiyama to be a completely positive map satisfying the property of the conditional expectation: $E(axb) = aE(x)b$ for any $a, b \in \mathfrak{A}_2$ and $x \in \mathfrak{A}_1$. A C^*-algebra $\mathfrak{A}$ with unit is called *injective* if any completely positive unit preserving linear map θ from $\mathfrak{A}$ into another

C^*-algebra $\mathfrak{B}$ with unit has an extension $\bar{\theta}$ to any C^*-algebra $\mathfrak{A}_1$ containing $\mathfrak{A}$ as a completely positive unit preserving linear map from $\mathfrak{A}_1$ into $\mathfrak{B}$. A von Neumann algebra M is injective if and only if it has Property E.

Effros and Lance called a von Neumann algebra M *semidiscrete* if the identity map from M into M is a weak pointwise limit of completely positive maps of finite rank and proved that M is injective if it is semi-discrete.

Connes [31] unified all these concepts by showing that they are all equivalent. The core result is the isomorphism of all injective factors of type II_1 to the unique hyperfinite factor R; it is established by a highly involved and technical proof, utilizing a theorem on tensor products of C^*-algebras, the property Γ of a factor which Murray and von Neumann introduced to distinguish some factors, properties of $\operatorname{Aut} N$ and $\operatorname{Int} N$ of a factor N, the ultra product R^ω for a free ultrafilter ω, an argument analogous to Day–Namioka proof of Følner's characterization of amenable groups, etc.

The uniqueness of injective factors of type II_1 then implies the uniqueness of injective factors of type II_∞. Together with an earlier uniqueness result for trace-scaling automorphisms of $R\otimes B(H)$ (exhibiting the unique injective factor of type II_∞), it also implies the uniqueness of injective factors of type III_λ, $0<\lambda<1$. With help of an earlier result of Krieger, injective factors of type III_0 are also completely classified by the isomorphism class of the so-called flow of weight. Thus Connes succeeded in a complete classification of AFD factors (which is as much as saying injective factors) except for the case of type III_1, which still remains open.

The work of Connes also shows that any continuous representation of a separable locally compact group G generates an injective von Neumann algebra if G/G_0 is amenable, where G_0 is the connected component of the identity (in particular, if G is connected or amenable).

5. Other works

After his success in the almost complete classification of injective factors, Connes turned his attention to application of operator algebras to differential geometry. Connes developed a non-commutative integration theory, which provides a method of integration over a family of ergodic orbits or over the set of leaves of a foliation. One significant outcome of this theory is an index theorem for foliation. I am sure that this subject will rapidly develop much further. For survey of the present status, we refer to [44], [50].

The works on positive cones [13] provide a geometric characterization of von Neumann algebras through the associated natural positive cone in the Hilbert space and lead to some applications.

A work connected with Kazdan's property T [42] provides a simple example of continuously many non-isomorphic factors of type II, and answers a question of Murray and von Neumann about the fundamental group of a factor of type II_1.

I hope that I have conveyed to you some feeling about the incredible power of Alain Connes and the richness of his contributions.

References

Published works of Alain Connes

[1] Sur une généralisation de la notion de corps ordonné, *C. R. Acad. Sci. Paris Sér A–B* **269** (1969), pp. A337–A340.

[2] Ordres faibles et localisation des zéros de polynômes, *C. R. Acad. Sci. Paris Sér. A–B* **269** (1969), pp. A373–A376.

[3] Ordres faibles et localisation de zéros de polynômes, *Séminaire Choquet: 1968/69, Initiation à l'Analyse,* Exp. 5, 27 pp. Secrétariat mathématique, Paris, 1969.

[4] Détermination de modèles minimaux en analyse non standard et application, *C. R. Acad. Sci. Paris Sér. A–B* **271** (1970), pp. A969–A971.

[5] Ultrapuissances et applications dans le cadre de l'analyse non standard, *Séminaire Choquet: 1969/70, Initiation à l'Analyse,* Fasc. 1, Exp. 8, 25 pp. Secrétariat mathématique, Paris, 1970.

[6] Un nouvel invariant pour les algèbres de von Neumann, *C. R. Acad. Sci. Paris Sér. A* **273** (1971), pp. 900–903.

[7] Calcul des deux invariants d'Araki–Woods par la théorie de Tomita–Takesaki, *C. R. Acad. Sci. Paris Sér. A* **274** (1972), pp. 175–177.

[8] États presque périodiques sur une algèbre de von Neumann, *C. R. Acad. Sci. Paris Sér. A* **274** (1972), pp. 1402–1405.

[9] Groupe modulaire d'une algèbre de von Neumann, *C. R. Acad. Sci. Paris Sér. A* **274** (1972), pp. 1923–1926.

[10] Une classification des facteurs de type III, *C. R. Acad. Sci. Paris Sér. A* **275** (1972), pp. 523–525.

[11] (Co-author: A. Van Daele) The Group Property of Invariant *S* of von Neumann Algebras, *Math. Scand.* **32** (1973), pp. 187–192.

[12] Une classification des fracteurs de type III, *Ann. Sci. Ecole Norm. Sup. 4ème Sér.* **6** (1973), pp. 133–252.

[13] Caractérisation des algèbres de von Neumann comme espaces vectoriels ordonnés, *Ann. Inst. Fourier* (Grenoble) **26** (1974), pp. 121–155.

[14] Almost Periodic States and Factors of type III_1, *J. Functional Analysis* **16** (1974), pp. 415–445.

[15] (Co-author: E. J. Woods) Existence de facteurs infinis asymptotiquement abéliens, *C. R. Acad. Sci. Paris Sér. A* **279** (1974), pp. 189–191.

[16] Sur le théorème de Radon–Nikodym pour les poids normaux fidèles semi-finis, *Bull. Sci. Math. 2ème Sér.* **97** (1973), pp. 253–258.

[17] On Hyperfinite Factors of Type III_0 and Krieger's Factors, *J. Functional Analysis* **18** (1975), pp. 318–327.

[18] (Co-author: E. Størmer) Entropy for Automorphisms of Finite von Neumann Algebras, *Acta Math.* **134** (1975), pp. 289–306.

[19] On the Hierarchy of W. Krieger, *Illinois J. Math.* **19** (1975), pp. 428–432.

[20] (Co–author: M. Takesaki) The Flow of Weights on Factors of Type III, *Tōhoku Math. J.* **29** (1977), pp. 473-575.

[21] (Co-author: M. Takesaki) Flot des poids sur les facteurs de type III, *C. R. Acad. Sci. Paris Sér. A* **278** (1974), pp. 945–948.

[22] A Factor not Antiisomorphic to Itself, *Bull. London Math. Soc.* **7** (1975) pp. 171–174.

[23] A Factor not Antiisomorphic to Itself, *Ann. Math.* (2) **101** (1975), pp. 536–554.

[24] Periodic Automorphisms of the Hyperfinite Factor of Type II_1, *Acta Sci. Math.* (Szeged) **39** (1977), pp. 39–66.

[25] The Tomita–Takesaki Theory and Classification of Type III Factors. In: D. Kastler (ed.), *Proceedings of the International School of Physics E. Fermi, Course* **60**, pp. 29–46.

[26] (Co-authors: P. Ghez, R. Lima, D. Testard and E. J. Woods) *Review of Crossed Product of von Neumann Algebras.* Mimeographed notes (unpublished).

[27] Structure Theory for Type III Factors. In: *Proceedings of the International Congress of Mathematicians*, vol. 2 (1975), pp. 87–91.

[28] Classification of Automorphisms of Hyperfinite Factors of Type II_1 and II_∞ and Application to Type III Factors, *Bull. Amer. Math. Soc.* **81** (1975), pp. 1090–1092.

[29] Outer Conjugacy Classes of Automorphisms of Factors, *Ann. Sci. École Norm. Sup. 4ème Sér.* **8** (1975), pp. 383–419.

[30] Sur la classification des facteurs de type II, *C. R. Acad. Sci. Paris Sér. A* **281** (1975), pp. 13–15.

[31] Classification of Injective Factors, *Ann. Math.* **104** (1976), pp. 73–115.

[32] (Co-author: E. Størmer) Homogeneity of the State Space of Factors of Type III_1, *J. Functional Analysis*, **28** (1978), pp. 187–196.

[33] On the Cohomology of Operator Algebras, *J. Functional Analysis* **28** (1978), pp. 248–253.

[34] (Co-author: W. Krieger) Measure Space Automorphisms, the Normalizers of Their Full Groups and Approximate Finiteness, *J. Functional Analysis* **24** (1977), pp. 336–352.

[35] (Co-author: W. Krieger) *Outer Conjugacy of Non-Commutative Bernoulli Shifts.* Mimeographed notes (unpublished).

[36] The Tomita–Takesaki Theory and Classification of Type III Factors. In: D. Kastler (ed.), *C*-algebras and Their Applications to Statistical Mechanics and Quantum Field Theory*, Proc. Int. School of Physics "Enrico Fermi", Varenna, 1973, Ital. Phys. Soc., 1976, pp. 29–46.

[37] The von Neumann Algebra of a Foliation. In: *Mathematical Problems in Theoretical Physics*, Lecture Notes in Phys. 80, Springer Verlag, 1978, pp. 145–151.
[38] Outer Conjugacy of Automorphisms of Factors, *Symposia Math.* Vol. 20, Academic Press, 1976, pp. 149–159.
[39] On the Equivalence Between Injectivity and Semi-Discreteness for Operator Algebra. In: *Colloq. Int. CNRS* no. 274 (Marseille, 1977), 1979, pp. 107–112.
[40] Von Neumann Algebras. In: O. Lehto (ed.), *Proceedings of the International Congress of Mathematicians, Helsinki 1978*, Acad. Sci. Fennica, 1980, pp. 97–109.
[41] (Co-author: E. J. Woods) A Construction of Approximately Finite-Dimensional non-ITPFI factors, *Canad. Math. Bull.* **23** (1980), pp. 227–230.
[42] A factor of Type II_1 with Countable Fundamental Group, *J. Operator Theory* **4** (1980), pp. 151–153.
[43] On the Spatial Theory of von Neumann Algebras, *J. Functional Analysis* **35** (1980), pp. 153–164.
[44] Sur la théorie non commutative de l'integration. In: P. de la Harpe (ed.), *Algèbres d'Operateurs*, Lecture Notes in Math. 725, Springer, 1979, pp. 19–143.
[45] Feuilletages et algèbres d'operateurs, *Séminaire Bourbaki 1979/80*, n° 551, Lecture Notes in Math. 842 (1981), pp. 139–155.
[46] C^*-algèbres et géométrie differentielle, *C. R. Acad. Sc. Paris* **290** A (1980), pp. 599–604.
[47] An Analogue of the Thom Isomorphism for Cross Products of a C^*-algebra by an Action of $\boldsymbol{R}$, *Adv. in Math.* **39** (1981), pp. 31–55.
[48] (Co-author: H. Moscovici) The L^2-Index Theorem for Homogeneous Spaces of Lie Groups, *Ann. of Math.* **115** (1982), pp. 291–330.
[49] (Co-authors: J. Feldmann and B. Weiss) Amenable Equivalence Relations Are Generated by a Single Transformation, *Ergodic Theory Dynamical Systems* **1** (1981), pp. 431–450.
[50] A Survey of Foliations and Operator Algebras, *Proceedings of Symposia in Pure Mathematics* (AMS), **38** (1982), Part 1, pp. 521–628.
[51] (Co-author: G. Skandalis) Theorème de l'indice pour les feuilletages, *C. R. Acad. Sci. Paris, Sér.* I, **292** (1981), pp. 871–876.
[52] (Co-author: V. Jones) A II_1 Factor with Two Nonconjugate Cartan Subalgebras, *Bull. Amer. Math. Soc.* **6** (1982), pp. 211–212.
[53] (Co-author: B. Weiss) Property T and Asymptotically Invariant Subspaces, *Israel J. Math.* **37** (1980), pp. 209–210.
[54] Spectral Sequence and Homology of Currents for Operator Algebras, *Faßungsbericht 42/81*, Oberwolfach (1981), pp. 4–5.
[55] (Co-author: G. Skandalis) *The Longitudinal Index Theorem for Foliations*, IHES Preprint M/82/24 (1982).
[56] Classification des facteurs. In: *Proceedings of Symposia in Pure Mathematics* (AMS), **38** (1982), Part 2, pp. 43–109.
[57] (Co-author: H. Moscovici) L^2-Index Theory on Homogeneous Spaces and Discrete Series Representations, In: *Proceedings of Symposia in Pure Mathematics* (AMS), **38** (1982), Part 1, pp. 419–433.

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES
KYOTO UNIVERSITY
KYOTO 606, JAPAN