A Sequence of Series for The Lambert W Function

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Abstract

We give a uniform treatment of several series expansions for the Lambert W function, leading to an infinite family of new series. We also discuss standardization, complex branches, a family of arbitrary-order iterative methods for computation of W, and give a theorem showing how to correctly solve another simple and frequently occurring nonlinear equation in terms of W and the unwinding number.

1 Introduction

Investigations of the properties of the Lambert W function are good examples of nontrivial interactions between computer algebra, mathematics, and applications. To begin with, the standardization of the name W by computer algebra (see section 1.2 below) has had several effects. First, this standardization has exposed a great variety of applications; second, it has uncovered a significant history, hitherto unnoticed because the lack of a standard name meant that most researchers were unaware of previous work; and, third, it has now stimulated current interest in this remarkable function. Further, many of the recent investigations have been carried out themselves using computer algebra, often forcing further development in computer algebra. This process has not yet reached its final state with regard to W.

Series expansions for W about various points play important roles in many studies and applications of W. Most basically they give starting values for numerical computation of W; further, they were one of the main factors in deciding the branch cuts for W; and in combinatorial applications series about the origin and about the branch point are important. Although several series for W have already been published, it has not been realized that there are systematic ways of developing them.

We here give a uniform treatment of several series expansions for W (sections 2.3, 2.4, 4.4, 4.6, 4.8 and 4.11 in particular contain results not previously published). We believe these results to be of direct interest for computer algebra researchers, partly because of their combinatorial applications but also because of their intrinsic value. A further point of interest to computer algebraists is that in order to represent these infinite series in Maple it is necessary to extend its knowledge of several special families of numbers (specifically Stirling cycle and subset numbers, associated Stirling numbers, and second-order Eulerian numbers).

We also present some infinite products for $W(z)$, some sequences converging to $W(z)$, the Laplace transform of $W(\exp(1+z))$ and the Mellin transform of $W(z)$.

Intended for ISSAC '97, Maui

1.1 Definitions

A review of the history, theory and applications of the Lambert W function may be found in [5]. The many-valued function $W(z)$ is defined as the root of

$$W(z)e^{W(z)} = z\,. \tag{1}$$

One may issue the Maple command

```
> plot( [ w*exp(w), w, w=-4..1] );
```

to see a graph of $W(x)$ for real x. For $x \geq 0$ there is only one real branch, but for $-1/e \leq x < 0$ there are two. Complex branches are discussed briefly in section 1.3 of this paper.

In this paper we also discuss series for the Tree function $T(z) = -W(-z)$, which satisfies

$$T(z)e^{-T(z)} = z\,. \tag{2}$$

This form of the function often occurs in combinatorial applications.

1.2 A note on notation

The notation $W(z)$ and the name 'the Lambert W function' have quickly become a standard, since the introduction of W into Maple sometime in the 1980's and the publication of [5] in its technical report form (January 1993). As of Maple V Release 4 the name of the function in Maple is `LambertW`. The reasons for naming the function after Lambert are detailed in [5]. Briefly, they are, first, that Euler clearly acknowledged Lambert in his paper "On a series of

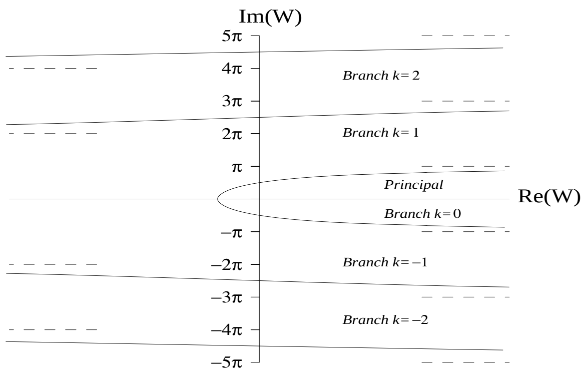

Figure 1: The ranges for the branches of the W function. Closure information is not indicated on the figure. Branches are closed on their top boundaries. This means in particular that only branch 0 and branch -1 take on real values. The curves other than the semi-line $(-\infty, -1]$ are described parametrically by $-\eta\cot\eta + i\eta$, $-\infty < \eta < \infty$. Incidentally, they form a subset of the curves called 'The Quadratrix of Hippias' which can be used both to square the circle and trisect the angle.

Lambert's and some of its significant properties" [9], where Euler developed the Taylor series for $-W(-z)$ and identified its radius of convergence, and, second, that naming yet another function after Euler would not be useful.

The use of the letter W was more or less accidental in Maple but turns out to have some significance in that E. M. Wright was the first person to do significant work on the complex branches of this function, and used it both in dynamical and combinatorial applications.

In the interests of true standardization, we suggest that any CAS implementing the function satisfying (1) use the name `LambertW` (with arguments `z` or `k,z`) and the branch cuts and closures of [5]. For documentation and for use in papers we recommend the notations $W(z)$ and $W_k(z)$.

We recognize that some designers may wish instead (or in addition) to implement the function satisfying (2), because the minus signs in the series (6) below are a distraction in combinatorial applications. Use of a different name and notation is perhaps justified in this case; we suggest $T(z)$, $T_k(z)$ (with the branches defined by $T_k(z) = -W_k(-z)$), `TreeT(z)`, and `TreeT(k,z)`. Note that DEK has suggested that Maple implement this function in addition to `LambertW` but this has not happened yet.

1.3 Complex branches of W

A detailed discussion of the complex branches of W can be found in [5]. The branches are denoted $W_k(z)$ (the exact Maple V Release 4 syntax is `LambertW(k,z)`) with $W_0(z)$ being the principal branch, satisfying $W_0(0) = 0$ and $W_0(x) > 0$ for $x > 0$. The only other branch to have real values is $W_{-1}(x)$ which takes on values in $(-\infty, -1]$ for $x \in [-1/e, 0)$. The ranges of the branches are indicated in Figure 1.

We will usually deal with the principal branch in this paper, except where otherwise noted, and will write $W(z)$ for $W_0(z)$ when there is no possibility for confusion.

2 Taylor series

The Taylor series for $W(z)$ about $z = 0$ has been known since Euler's paper [9]. The series can be derived very simply using the Lagrange Inversion Formula (see e.g. [3]). However, it is of historical interest to note that Lambert's derivation of his series pre-dates the Lagrange Inversion Formula. We here use more modern tools, as expounded in the very elegant paper [17]. But before we give the full derivation, note that the first several terms may be computed very simply in a computer algebra system by the commands (here written in Maple for convenience)

```
> series(w*exp(w)-z, w);
```

$$-z + w + w^2 + \frac{1}{2}w^3 + \frac{1}{6}w^4 + \frac{1}{24}w^5 + O(w^6)\,.$$

```
> solve(", w);
```

$$z - z^2 + \frac{3}{2}z^3 - \frac{8}{3}z^4 + \frac{125}{24}z^5 + O(z^6)$$

To get the complete series, we note that if $f(w) = w\phi(w) = w\exp w$ then $\phi(w) = \exp w$ and

$$\phi(w)^s = e^{sw} = \sum_{k\ge 0} \frac{s^k}{k!} w^k\,. \tag{3}$$

This means that the *suite of polynomials of binomial type* [17] associated with $f(w)$ is just

$$P_k(s) = \frac{s^k}{k!}\,. \tag{4}$$

The formal series for $f^{-1}(z)$ is in general [17]

$$f^{-1}(z) = z\sum_{n\ge 0} \frac{P_n(-1-n)}{n+1} z^n \tag{5}$$

so in our case, by using our known $P_k(s)$ we get

$$W(z) = \sum_{n\ge 1} \frac{(-n)^{n-1}}{n!} z^n\,. \tag{6}$$

The ratio test establishes that this series converges if $|z| < 1/e$.

We might instead try to derive the Taylor series directly, since a formula for the n-th derivative of W for $n \ge 1$ is known [5, p. 340]. The formula is

$$\frac{d^n W(x)}{dx^n} = \frac{\exp(-nW(x))p_n(W(x))}{(1+W(x))^{2n-1}}\,, \tag{7}$$

where the polynomials $p_n(w)$ satisfy the recurrence relation

$$p_{n+1}(w) = -(nw + 3n - 1)p_n(w) + (1+w)p_n'(w)\,. \tag{8}$$

However, finding $p_n(0)$ from this recurrence is more difficult than the previous derivation. The initial polynomial is $p_1(w) = 1$. The next few polynomials are, by Maple, $p_2(w) = -2 - w$, $p_3(w) = 9 + 8w + 2w^2$, $p_4(w) = -64 - 36w^2 - 79w - 6w^3$, and $p_5(w) = 974w + 625 + 622w^2 + 192w^3 + 24w^4$. A simple consequence of the recurrence is that if $W(a) = w$ is rational, then the Taylor series of W about $x = a$ has rational coefficients, apart from the factor a^n which occurs if $a \ne 0$. This uses the fact that $\exp(nw) = a^n/w^n$.

It is clear that $W(a)$ will be rational if and only if $a = r\exp(r)$ for some rational r. But W can take on other simple values, and this will be the subject of an upcoming paper by David Jeffrey and co-workers.

2.1 Some series for $T(z) = -W(-z)$

We have immediately that

$$T(z) = \sum_{n\ge 1} \frac{n^{n-1}}{n!} z^n \qquad (9)$$

but Lagrange inversion (or the suite of polynomials treatment) gives more:

$$T(z)^x = \sum_{n\ge 0} \frac{x(n+x)^{n-1} z^{n+x}}{n!} \qquad (10)$$

(cf. [14, exercise 2.3.4.4–29]). In particular, for integer k we have

$$T(z)^k = k \sum_{n\ge 1} \frac{n^{n-k}(n-1)\cdots(n-k+1)z^n}{n!} . \qquad (11)$$

This gives a connection with 'Q series' (see [14]): if

$$\begin{aligned} g(n) &= Q(a_1, a_2, a_3, \ldots; n) \\ &= a_1 + a_2 \frac{n-1}{n} + a_3 \frac{n-1}{n}\frac{n-2}{n} + \cdots , \end{aligned} \qquad (12)$$

then

$$a_1 T(z) + \tfrac{1}{2} a_2 T(z)^2 + \tfrac{1}{3} a_3 T(z)^3 + \cdots = \sum_{n\ge 1} \frac{g(n) n^{n-1} z^n}{n!} . \qquad (13)$$

This follows on substitution of equation (11) and rearranging. Since $Q(1,2,3,\ldots;n) = n$, an immediate consequence is

$$\frac{1}{1-T(z)} = \sum_{n\ge 0} \frac{n^n z^n}{n!} . \qquad (14)$$

Similarly, $Q(2\cdot 1^2, 3\cdot 2^2, 4\cdot 3^2, \ldots; n) = 2n^2$ and hence

$$\frac{T(z)}{(1-T(z))^3} = \sum_{n\ge 1} \frac{n^{n+1} z^n}{n!} . \qquad (15)$$

Note also that $(1+x/n)^{n-1} = Q(1, x/1!, x^2/2!, \ldots; n)$, which yields the identity

$$e^{xT(z)} = \sum_{n\ge 0} \frac{x(n+x)^{n-1} z^n}{n!} . \qquad (16)$$

We have essentially seen this before (in equation (10)), since $\exp(xT(z)) = (T(z)/z)^x$. In its new form it makes sense for complex x.

Let $Q(n) = Q(1,1,1,\ldots;n)$ be Ramanujan's function

$$Q(n) = 1 + \frac{n-1}{n} + \frac{n-1}{n}\frac{n-2}{n} + \cdots . \qquad (17)$$

Then

$$\ln \frac{1}{1-T(z)} = \sum_{n\ge 1} \frac{n^{n-1} Q(n) z^n}{n!} \qquad (18)$$

$$\frac{T(z)}{(1-T(z))^2} = \sum_{n\ge 1} \frac{n^n Q(n) z^n}{n!} \qquad (19)$$

$$\frac{T(z)+T(z)^2}{(1-T(z))^4} = \sum_{n\ge 1} \frac{n^{n+1} Q(n) z^n}{n!} . \qquad (20)$$

The number of mappings from $\{1,2,\ldots,n\}$ into itself having exactly k component cycles is the coefficient of y^k in $t_n(y)$ where $t_n(y)$ is the *tree polynomial* of order n (see [16]) and is generated by

$$\frac{1}{(1-T(z))^y} = \sum_{n\ge 0} t_n(y) \frac{z^n}{n!} . \qquad (21)$$

Comparison with (18) gives for $n \ge 1$

$$\lim_{y\to 0} \frac{t_n(y)}{y} = t_n'(0) = \frac{Q(n) n^{n-1}}{n!} . \qquad (22)$$

These series and others may be found in [16], where they are used to analyze a recurrence related to trees.

Finally, note that using equation (13) with

$$\begin{aligned} g(n) &= Q(1, 2\alpha, 3\alpha^2, \ldots; n) \\ &= F\left({1, 2, 1-n \atop 1} \middle| , \frac{-\alpha}{n} \right) \end{aligned} \qquad (23)$$

where the hypergeometric function F is written using the notation of [10], allows us to write series expansions for any rational function of $T(z)$, by first expanding in partial fractions.

2.2 Taylor series for $W(\exp x)$

The equation

$$y + \ln y = z \qquad (24)$$

often occurs in applications. Its solution in terms of W is (ignoring branches for the moment) $y = W(\exp z)$, which in some ways is a nicer function than $W(z)$. For one thing, its derivatives are slightly simpler, and without much difficulty one can establish by induction that for $n \ge 1$

$$\frac{d^n W(e^z)}{dz^n} = \frac{q_n(W(e^z))}{(1+W(e^z))^{2n-1}} , \qquad (25)$$

where the polynomials $q_n(w)$ are given by

$$q_n(w) = \sum_{k=0}^{n-1} \left\langle\!\!\left\langle {n-1 \atop k} \right\rangle\!\!\right\rangle (-1)^k w^{k+1} . \qquad (26)$$

These polynomials have coefficients expressed in terms of second-order Eulerian numbers [10]. See section 4.1 for recurrence relations for these numbers. The $q_n(w)$ may be computed from the recurrence relation [5]

$$q_{n+1}(w) = -(2n-1) w q_n(w) + (w + w^2) q_n'(w) . \qquad (27)$$

Discovering formulas (25–26) in the first place is another matter, of course, and it is here that computation of the first several derivatives can point the way.

```
> alias(  w = LambertW( exp(z) )  );
> seq( normal(diff( w, z$n)), n=1..5);
```

$$[\frac{w}{1+w}, \frac{w}{(1+w)^3}, -\frac{w\,(-1+2\,w)}{(1+w)^5}, \frac{w\,(1-8\,w+6\,w^2)}{(1+w)^7}, -\frac{w\,(-1+22\,w-58\,w^2+24\,w^3)}{(1+w)^9}]$$

As before, when $w = W(\exp a)$ is rational the Taylor series for $W(\exp z)$ about $z = a$ has rational coefficients. For example, this occurs when $a = 1$, giving

$$W(e^z) = 1 + \frac{1}{2}(z-1) + \frac{1}{16}(z-1)^2 - \frac{1}{192}(z-1)^3 - \frac{1}{3072}(z-1)^4 + \frac{13}{61440}(z-1)^5 + O((z-1)^6)\,. \qquad (28)$$

This series has radius of convergence $\sqrt{4+\pi^2}$, which is the distance to the nearest singularity at $z = -1 + i\pi$. This gives the asymptotics of $q_n(1)$.

2.3 Branches in equation (24)

Theorem: The unique solution of $y + \ln y = z$ is

$$y = W_{-\mathcal{K}(z)}(e^z) \qquad (29)$$

where $\mathcal{K}(z)$ is the unwinding number of z (see [6]), unless $z = t + i\pi$ for $-\infty < t \le -1$, in which case there are exactly two solutions, $y = W_{-1}(\exp z)$ and $y = W_0(\exp z)$.

Proof. Taking exponentials of both sides of $y + \ln y = z$ we see that if y is a solution, then $y = W_k(\exp z)$ for some k. To go in the other direction, we use the relation

$$W_k(z) + \ln W_k(z) = \ln z + 2\pi i k \qquad (30)$$

unless $k = -1$ and $z \in [-1/e, 0)$, when $W_{-1}(z) + \ln W_{-1}(z) = \ln z$. For a proof of this relation see [12]. We replace z in the above by $\exp z$, and since

$$\ln e^z = z + 2\pi i \mathcal{K}(z) \qquad (31)$$

(indeed this defines the unwinding number $\mathcal{K}(z)$, see [6]), we have that

$$W_k(e^z) + \ln W_k(e^z) = z + 2\pi i\left(\mathcal{K}(z) + k\right) \qquad (32)$$

unless $k = -1$ and $\exp z \in [-1/e, 0)$, in which case we replace k by 0 on the right hand side of (32). Thus we have that $W_k(\exp z) + \ln W_k(\exp z) = z$ if and only if $k = -\mathcal{K}(z)$, unless $k = -1$, as claimed. It is easy to see that both $k = -1$ and $k = 0$ work if $z = t + i\pi$ for some $t \in (-\infty, -1]$ as claimed, and because the unwinding numbers for $z = t + i(2m+1)\pi$ are all different from 0 if $m \neq 0$, this half-line is the only set of exceptions.

Remark. It is very interesting that we may write the simple formula (29) for a (nearly) single-valued function $y(z)$ in terms of a (very) multi-valued function W_k, using another simple function of z to deftly switch branches as required. This clean formulation, valid except on a single half-line in the z-plane, confirms that the branch choices made in [5] (which were done so some asymptotic series worked out nicely) were convenient ones.

2.4 Laplace and Mellin transforms

The series (28) is quite remarkably connected with the asymptotic expansion for 'Airey's convergence factor' (which was used in [20] to improve convergence of the asymptotic series for the exponential integral. See [2] for details). This is essentially because the Laplace transform of $W(\exp z)$ can be evaluated in terms of the incomplete Gamma function as follows:

$$\begin{aligned}\mathcal{L}\left(W(e^{1+z})\right) &= \int_0^\infty e^{-sz} W(e^{1+z})\,dz \\ &= \frac{1}{s} + e^s s^{s-2}\Gamma(1-s, s)\,. \end{aligned} \qquad (33)$$

Getting Maple to evaluate this integral requires explicit use of `changevar` and assumptions on s. The requisite substitution is $w = W(\exp(1+z))$ so $dz = (1+w)dw/w$ and $\exp z = w\exp(w-1)$.

The effect of the Laplace transform, as can be verified with Watson's lemma [1], is to convert an exponential generating function into an ordinary generating function (in $1/s$; one can keep the same form of the power series by using the Borel transform $\int_0^\infty \exp(-t) f(st)\,dt$, but this is only trivially different).

Since we can evaluate the Laplace transform of $W(\exp z)$ it is no surprise that we can evaluate the Mellin transform of $W(z)$. The result is

$$\begin{aligned} M(W(z)) &= \int_0^\infty x^{s-1} W(x)\,dx \\ &= \frac{(-s)^{-s}\Gamma(s)}{s}\,, \end{aligned} \qquad (34)$$

for $-1 < \Re(s) < 0$. Again explicit use of `changevar` and assumptions are necessary to get Maple to evaluate this transform.

3 Series about the branch point

If we put $p = \sqrt{2(ez+1)}$ in $W \exp W = z$, and expand in powers of $1 + W$ and revert, we obtain

$$W_0(z) = \sum_{\ell \ge 0} \mu_\ell p^\ell = -1 + p - \tfrac{1}{3}p^2 + \tfrac{11}{72}p^3 + \cdots\,. \qquad (35)$$

This series is well-known and has many combinatorial applications (see for example [16]). It converges for $|p| < 2^{1/2}$. The coefficients may be computed by the following recurrence relations, which were communicated to us by Don Coppersmith.

$$\begin{aligned} \mu_k &= \frac{k-1}{k+1}\left(\tfrac{1}{2}\mu_{k-2} + \tfrac{1}{4}\alpha_{k-2}\right) - \tfrac{1}{2}\alpha_k - \tfrac{1}{k+1}\mu_{k-1} \\ \alpha_k &= \sum_{j=2}^{k-1} \mu_j \mu_{k+1-j}\,, \qquad \alpha_0 = 2, \qquad \alpha_1 = -1\,, \end{aligned} \qquad (36)$$

where $\mu_0 = -1$ and $\mu_1 = 1$.

If $\Im(z) \ge 0$ we may take $p = -\sqrt{2(ez+1)}$ in this series to get a good approximation to $W_{-1}(z)$. If instead $\Im(z) < 0$, then the series (with the negative sign on p) gives a good approximation to $W_1(z)$. This 'branch splitting' is a result of the branch choices for W, and is the price we pay for convenient asymptotic expansions at 0 and ∞.

3.1 Branch point series for $W(\exp z)$

The paper [19] makes a delightful connection between Stirling's approximation for $n!$ and W. That there is a connection was pointed out to RMC and DJJ some years ago by Bruno Salvy, who said in an e-mail:

> It is possible to use this to prove Stirling's formula. One first computes the local expansion of W at its singularity by

```
> op({solve(series(subs(w=-1+u, z=-exp(-1)
+ t^2* exp(-1),w*exp(w)-z),u),u)}) ;
```

$$\sqrt{2}\,t - \frac{2}{3}\,t^2 + \frac{11}{36}\,\sqrt{2}\,t^3 - \frac{43}{135}\,t^4 + \frac{769}{4320}\,\sqrt{2}\,t^5 - \frac{1768}{8505}t^6 + O(t^7)\,,$$
$$-\sqrt{2}\,t - \frac{2}{3}\,t^2 - \frac{11}{36}\,\sqrt{2}\,t^3 - \frac{43}{135}\,t^4 - \frac{769}{4320}\,\sqrt{2}\,t^5 - \frac{1768}{8505}\,t^6 + O(t^7)$$

where $t = (1+ez)^{1/2}$. Now, since the singularity is isolated, it is possible to use Darboux's theorem which says that the asymptotic expansion of the n-th Taylor coefficient of W at the origin is obtained by taking the coefficients term by term in the above equation. In other words (considering only the first order term):

$$\frac{(-n)^n}{n!} = \frac{(-e)^n}{\sqrt{2\pi n}} \qquad (37)$$

which is Stirling's formula. [...] I heard it a few years ago from Philippe Flajolet.

The connection between series at the branch point for W and Stirling's formula for $n!$ is made explicit in [19], who use elementary arguments, and we paraphrase their results below. Starting from $n! = \int_0^\infty x^n \exp(-x)\,dx$, changing variables to $y = x/n$ and splitting the range of integration into $[0,1]$ and $[1,\infty]$, they get

$$n! = n^{n+1}e^{-n}\left[\int_0^1 (ue^{1-u})^n\,du + \int_1^\infty (ve^{1-v})^n\,dv\right]\,. \qquad (38)$$

They then make *ad hoc* substitutions; here we use our knowledge of W and its branches to repeat their work. Putting

$$u = T_0\left(e^{-1-z^2/2}\right) \quad \text{and} \qquad (39)$$
$$v = T_{-1}\left(e^{-1-z^2/2}\right)\,, \qquad (40)$$

we arrive at

$$n! = n^{n+1}e^{-n}\int_0^\infty ze^{-nz^2/2}\left[\frac{1}{1-T_{-1}} - \frac{1}{1-T_0}\right]dz\,. \qquad (41)$$

Note that $z = 0$ corresponds to the branch point of T. We use the T notation to keep minus signs to a minimum. We now expand the expression in brackets about $z = 0$ (it is more convenient to use the series for du and dv directly, as was done in [19], than to rearrange the geometric series for $1/(1-T)$ in (41)), integrate term by term, and arrive at the asymptotic expansion for $n!$ in the form

$$n! = n^{n+1}e^{-n}\sum_{k\geq 0}(2k+1)a_{2k+1}\left(\frac{2}{n}\right)^{k+1/2}\Gamma(k+1/2) \qquad (42)$$

where

$$T_{-1}(e^{-1-z^2/2}) = \sum_{n\geq 0} a_n z^n\,, \qquad (43)$$

and the series for T_0 is the same but with alternating signs. This gives at last that

$$n! = \sqrt{2\pi n}n^n e^{-n}\sum_{k\geq 0}\frac{1\cdot 3\cdot 5\cdots(2k+1)}{n^k}a_{2k+1}\,. \qquad (44)$$

The following Maple procedure implements the recurrence relation derived in [19] for the a_k.

```
a := proc(n) option remember;
  local k;
  1/(n+1)/a(1)*(a(n-1) -
  add( k*a(k)*a(n+1-k), k=2..n-1))
end:
a(0) := 1;
a(1) := 1;
```

The choice $a(1) = 1$ gives the series for v, whilst $a(1) = -1$ gives the series for u. In more mathematical notation, their recurrence relation is

$$a_0 = 1 \qquad (45)$$
$$a_1 = \pm 1 \qquad (46)$$
$$a_n = \frac{1}{(n+1)a_1}\left(a_{n-1} - \sum_{k=2}^{n-1} ka_k a_{n+1-k}\right)\,. \qquad (47)$$

The first few terms are $v = T_{-1}(e^{-1-z^2/2}) =$

$$1 + z + \frac{1}{3}\,z^2 + \frac{1}{36}\,z^3 - \frac{1}{270}\,z^4 + \frac{1}{4320}\,z^5 + \frac{1}{17010}\,z^6 - \frac{139}{5443200}\,z^7 + \frac{1}{204120}\,z^8 - \frac{571}{2351462400}\,z^9 - \frac{281}{1515591000}\,z^{10} + \frac{163879}{2172751257600}\,z^{11} + O(z^{12})\,. \qquad (48)$$

Since the nearest other singularity is at $z = \infty$, this series converges for all finite z.

3.2 Other branch point series

The relation between W_{-1} and W_0 near the branch point was studied by Karamata in [13], who studied the coefficients in the power series

$$\mu = \sigma + \tfrac{2}{3}\sigma^2 + \tfrac{4}{9}\sigma^3 + \cdots = \sum_{n\geq 1} c_n\sigma^n\,, \qquad (49)$$

μ being the solution to

$$(1+\mu)e^{-\mu} = (1-\sigma)e^{\sigma}\,. \qquad (50)$$

In terms of T, the solutions are

$$1+\mu = \begin{cases} T_0\left((1-\sigma)e^{-(1-\sigma)}\right) \\ T_{-1}\left((1-\sigma)e^{-(1-\sigma)}\right)\end{cases} \qquad (51)$$

where if $\sigma > 0$ then T_0 simplifies to $1-\sigma$ and gives $\mu = -\sigma$ but T_{-1} does not simplify, whilst if $\sigma < 0$ then T_{-1} simplifies to $1-\sigma$ but T_0 does not. Karamata has developed a series for the root that does not simplify. Using the differential equation

$$\mu\frac{d\mu}{d\sigma} = \left(1 - \frac{1}{1-\sigma}\right)(1+\mu) \ , \tag{52}$$

it is easy to derive the recurrence relation

$$c_n = \frac{1}{n+1}\left[2 + \sum_{j=2}^{n-1} c_j(1 - jc_{n-j+1})\right] \ , \tag{53}$$

which is valid for $n \geq 2$ if the sum is taken to be 0 if empty.

An interesting parametric description of the real branches of W is contained in [18]. If we put

$$p = \tfrac{1}{2}\left(W_0(z) - W_{-1}(z)\right) \ , \tag{54}$$

then [18] gives

$$\begin{aligned} z &= -pe^{-p\coth p}\operatorname{cosech} p & (55)\\ W_0(z) &= -pe^{p}\operatorname{cosech} p & (56)\\ W_{-1}(z) &= -pe^{-p}\operatorname{cosech} p & (57)\end{aligned}$$

and expands these in series containing Bernoulli numbers.

The connection with Karamata's work is that

$$p = (\sigma + \mu)/2 \ , \tag{58}$$

though it is not clear at this writing if this gives us any new information.

4 Series about Infinity

We collect and present the previously known series for $W(z)$ about infinity, which go back to de Bruijn [8] and to Comtet [4]. We then show that these known series are members of an infinite family of series (some of these were exhibited in [5, 11]), and show that they are valid as $z \to 0$ as well as for $z \to \infty$. Finally we exhibit some new series, including an expansion for $W(\rho\exp(it))$ in terms of $W(\rho)$ and it which separates the real and imaginary parts of $W(\rho\exp(it))$.

4.1 Stirling and Eulerian numbers

To fix notation, we summarize with generating functions. *Stirling cycle numbers* $\left[{n \atop m}\right]$ are generated (for example) by [10]

$$\ln^m(1+z) = m!\sum_{n\geq 0}(-1)^{n+m}\left[{n \atop m}\right]\frac{z^n}{n!} \ . \tag{59}$$

The numbers $(-1)^{n+m}\left[{n \atop m}\right]$ are also called Stirling numbers of the first kind. See [15] for a discussion of the advantages of the notation used here.

The *Stirling subset numbers* $\left\{{n \atop m}\right\}$, which are also called Stirling numbers of the second kind, are generated (for example) by

$$(e^z - 1)^m = m!\sum_{n\geq 0}\left\{{n \atop m}\right\}\frac{z^n}{n!} \ . \tag{60}$$

Recurrence relations (and much more) for these numbers may be found in [10].

We also make use of r-associated Stirling subset numbers (we use only the case $r = 2$), which are generated by

$$(e^z - \sum_{k=0}^{r-1}\frac{z^k}{k!})^m = m!\sum_{n\geq 0}\left\{{n \atop m}\right\}_{\geq r}\frac{z^n}{n!} \ . \tag{61}$$

Finally, we need the second-order Eulerian numbers. As before we follow [10]. The numbers are defined as follows.

$$\begin{aligned} \left\langle\!\!\left\langle {n \atop n} \right\rangle\!\!\right\rangle &= \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise}\end{cases} & (62)\\ \left\langle\!\!\left\langle {n \atop 0} \right\rangle\!\!\right\rangle &= 1 & (63)\\ \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle &= (k+1)\left\langle\!\!\left\langle {n-1 \atop k} \right\rangle\!\!\right\rangle \\ &\quad + (2n-1-k)\left\langle\!\!\left\langle {n-1 \atop k-1} \right\rangle\!\!\right\rangle . & (64)\end{aligned}$$

4.2 The fundamental relation

If we start with $W(z)\exp(W(z)) = z$, then by putting $W(z) = \ln z - \ln\ln z + u$ (which defines u for any $z \neq 0$, but which makes particularly good sense if z is large) we get

$$(\ln z - \ln\ln z + u)\frac{z}{\ln z}e^u = z \tag{65}$$

and since $z \neq 0$ we may divide both sides by z, and put $\sigma = 1/\ln z$ and $\tau = \ln\ln z/\ln z$ to get

$$1 - \tau + \sigma u = e^{-u} \ . \tag{66}$$

This is what we call the fundamental relation, and we will see some of its consequences in what follows. Note that this same equation, *mutatis mutandis*, arises if we replace $W_k(z)$ by $\ln_k z - \ln\ln_k z + u^{(k)}$, where $\ln_k(z) = \ln(z) + 2\pi ik$ is the notation introduced in [12] for the k-th logarithm branch.

4.3 de Bruijn and Comtet's expansion

de Bruijn solved the fundamental equation (66) in series to show that the asymptotic expansion of $W(x)$ for large x is in fact convergent [8]. Comtet later identified the coefficients explicitly as Stirling numbers [4]. The result is (explicitly adding branch information which Comtet did not use, and specifically excluding the case where $k = -1$ and $z \in [-1/e, 0)$ where another expansion holds)

$$\begin{aligned} W_k(z) &= \ln_k(z) - \ln\ln_k(z) \\ &\quad + \sum_{\ell\geq 0}\sum_{m\geq 1} c_{\ell m}\frac{(\ln\ln_k(z))^m}{(\ln_k z)^{\ell+m}} \ , & (67)\end{aligned}$$

where $c_{\ell m} = \frac{1}{m!}(-1)^\ell\left[{\ell+m \atop \ell+1}\right]$ is expressed in terms of Stirling cycle numbers. This (absolutely convergent for large enough z) doubly infinite series can then be rearranged to get

$$\begin{aligned} W_k(z) &= \ln_k z - \ln\ln_k z & (68)\\ &\quad + \sum_{n\geq 1}\frac{(-1)^n}{(\ln_k z)^n}\sum_{m=1}^{n}(-1)^m\left[{n \atop n-m+1}\right]\frac{(\ln\ln_k z)^m}{m!} \ . \end{aligned}$$

These series were further developed and rearranged in [11] using the new variable $\zeta = 1/(1+\sigma)$ to get

$$W_k(z) = \ln_k z - \ln\ln_k z + \sum_{m\ge 1} \frac{\tau^m}{m!} \sum_{p=0}^{m-1} (-1)^{p+m-1} \zeta^{p+m} \left\{ \begin{matrix} p+m-1 \\ p \end{matrix} \right\}_{\ge 2}, \tag{69}$$

and a further series which we will discuss below.

4.4 A new series

The fundamental equation (66) may be solved for u, *for any σ and τ, not just the values used here*, in terms of W. The solution is

$$u = W\left(\frac{1}{\sigma} e^{(1-\tau)/\sigma}\right) - \frac{1-\tau}{\sigma}. \tag{70}$$

If $\sigma = 1/\ln z$ and $\tau = \ln\ln z/\ln z$ as we started with, then this equation gives no new information—apparently. Indeed, when simplified, equation (70) becomes

$$W(z) = W\left(\frac{1}{\sigma} e^{(1-\tau)/\sigma}\right) \tag{71}$$

which really only rewrites z using the new variables.

After some thought, we find that if, instead of following de Bruijn and using the variables σ and τ, we use the variables $v = 1/\sigma$ and $p = -\tau/\sigma$, giving

$$W(z) = W\left(\ln z e^{\ln z} e^{-\ln\ln z}\right) = W\left(v e^v e^p\right), \tag{72}$$

then our formulas become simpler and more comprehensible. We will also need the *unwinding number for W*, namely the function $\mathcal{U}_\ell(v)$ given by

$$W_\ell\left(v e^v\right) = v + 2\pi i \mathcal{U}_\ell(v). \tag{73}$$

One can show that $\mathcal{U}_\ell(v) = \mathcal{K}(v + \ln v) + \ell + O(1/v)$ is asymptotically an integer. Moreover, if v is in the range of W_ℓ, then $\mathcal{U}_\ell(v) = 0$.

Finally, we must extend the formula for the n-th derivative of $W(\exp z)$ to the following:

$$\left.\frac{d^n W(a e^a e^{bz})}{dz^n}\right|_{z=0} = \frac{b^n q_n(W(ae^a))}{(1 + W(ae^a))^{2n-1}}. \tag{74}$$

This gives us the series expansion

$$W(z) = \sum_{n\ge 0} \frac{q_n(v + 2\pi i \mathcal{U}(v))}{(1 + v + 2\pi i \mathcal{U}(v))^{2n-1}} \frac{p^n}{n!}. \tag{75}$$

Remark. This series provides a foundation for an infinite number of series for $W(z)$. We first observe the remarkable fact that if $v = \ln z$ and $p = -\ln\ln z$, then this series is asymptotic as $z \to \infty$ (and also convergent for large z), and also asymptotic as $z \to 0$ for all branches except the principal branch. That is, the same series works for small z and for large! Further, if $v = \ln(-x)$ and $p = \ln(-\ln(-x))$, then (75) also works for the exceptional case $W_{-1}(x)$ as $x \to 0^-$. Finally, we may fix z and consider the asymptotics of $W_k(z)$ as $|k|$ gets large. Using relation (30) as a guide, we put $v = \ln_k z$ and $p = -\ln\ln_k z$. We can show that the unwinding numbers $\mathcal{U}_k(v)$ are zero for large enough v, and that the terms in this series form an asymptotic sequence as $z \to \infty$ (essentially since the denominator is degree $2n-1$ in v while the numerator is degree $n-1$, and because the p^n terms grow more slowly). Thus this series again gives the correct asymptotics. These observations were used in section 4.3 to extend Comtet's series to arbitrary branches of W.

The following Maple session shows the first few terms of this new series (75).

```
> basic := 1 - tau + sigma*u - exp(-u)
```

$$\mathrm{basic} := 1 - \tau + \sigma\, u - e^{(-u)}$$

```
> subs(sigma=1/v,tau=-p/v,");
```

$$1 + \frac{p}{v} + \frac{u}{v} - e^{(-u)}$$

```
> series(",u);
```

$$\frac{p}{v} + \left(\frac{1}{v} + 1\right) u - \frac{1}{2} u^2 + \frac{1}{6} u^3 - \frac{1}{24} u^4 + \frac{1}{120} u^5 + O(u^6)$$

```
> solve(",u):
> series(v + p + ",p):
> W(z) = map(normal,");
```

$$W(z) = v + \frac{v}{1+v} p + \frac{1}{2} \frac{v}{(1+v)^3} p^2 - \frac{1}{6} \frac{v\,(-1+2\,v)}{(1+v)^5} p^3 + \frac{1}{24} \frac{v\,(6\,v^2 - 8\,v + 1)}{(1+v)^7} p^4 - \frac{1}{120} \frac{v\,(24\,v^3 - 58\,v^2 + 22\,v - 1)}{(1+v)^9} p^5 + O(p^6)$$

This is a truncation of (75) and as predicted it contains second-order Eulerian numbers.

4.5 Convergence

The series (75) converges, as a series in p, for all $|p|$ less than the distance to the singularity at $v\exp(v + \hat{p}) = -1/e$. Because $\exp(2\pi i k) = 1$, we have to find the nearest such singularity. The logarithmic singularity at $z = 0$ has been moved to $p = \infty$.

4.6 An infinite sequence of series

Experimentation with the series (75) suggested many rearrangements (some of which are given in section 4.9). But our experiments also suggested that equations (72) and (75) are invariant under the transformations

$$\begin{aligned} v_{n+1} &= v_n + p_n \\ p_{n+1} &= -\ln(1 + p_n/v_n). \end{aligned} \tag{76}$$

We take v_0 and p_0 to be the v and p already defined. This invariance was already observed in [11] (in terms of an iteration for σ and τ) but the full significance was not then completely understood. It is now clear that this iteration may be run forward, for $n \to \infty$, and backward, for $n \to -\infty$. Running the iteration backward gives

$$\begin{aligned} v_{n-1} &= v_n e^{p_n} \\ p_{n-1} &= v_n \left(1 - e^{p_n}\right). \end{aligned} \tag{77}$$

In the case $n \to \infty$, one can show that $v_n \to W(z)$ if $|W(z)| > 1$, while $v_n \to W(z)$ in $|W(z)| < 1$ if $n \to -\infty$.

In fact this iteration turns out to be completely equivalent to the well-studied exponential iteration defining

$$z^{z^{z^{\cdot^{\cdot^{\cdot}}}}} \tag{78}$$

and this connection is discussed in detail elsewhere [7]. For our purposes we note that this gives us an infinite number of series for $W(z)$, since

$$z = v_n e^{v_n} e^{p_n} \tag{79}$$

as can easily be established by induction. This gives us a family of bi-infinite sequences of series

$$W_\ell(z) = \sum_{k \ge 0} \frac{q_k(v_n + 2\pi i \mathcal{U}_\ell(v_n))}{(1 + v_n + 2\pi i \mathcal{U}_\ell(v_n))^{2k-1}} \frac{p_n^k}{k!} \tag{80}$$

for $W(z)$. We may add a further infinity of series by using different branches for ln, by putting $v_0^{(m)} = \ln_m(z)$ and $p_0^{(m)} = -\ln\ln_m(z)$. It is this sequence of series that gives the title of this paper.

If $|W(z)| > 1$ then $p_n/v_n \to 0$ and $v_n \to W(z)$ as $n \to \infty$, and hence these series converge faster for larger n. If instead $|W(z)| < 1$ then $p_n/v_n \to 0$ and $v_n \to W(z)$ as $n \to -\infty$, and hence these series converge faster for more negative n.

Remark. If $v_0 = \ln z$ and $p_0 = -\ln\ln z$, then $v_{-1} = 1$ and $p_{-1} = \ln(z/e)$. This makes (80) particularly simple, being a series of polynomials in $\ln(z/e)$. This gives

$$W(z) = 1 + \frac{1}{2}\ln(\frac{z}{e}) + \frac{1}{16}\ln(\frac{z}{e})^2 - \frac{1}{192}\ln(\frac{z}{e})^3 - \cdots \tag{81}$$

which is really just (75) with $z = 1 \cdot \exp(1)\exp(\ln(z) - 1)$.

A more interesting series comes from $v_{-2} = z/e$ and $p_{-2} = 1 - z/e$. This new series contains only terms rational in z, and is quite accurate for small and moderate z. The first few terms are

$$W(z) = \frac{z}{e} - \frac{(1 - \frac{e}{z})\, z^2}{e^2\,(1 + \frac{z}{e})} + \frac{1}{2}\,\frac{(1 - \frac{e}{z})^2\, z^3}{e^3\,(1 + \frac{z}{e})^3} - \frac{1}{6}\,\frac{(1 - \frac{e}{z})^3\, z^3\,(-2\,\frac{z^2}{e^2} + \frac{z}{e})}{e^3\,(1 + \frac{z}{e})^5} + \cdots \tag{82}$$

4.7 A completely different series

Using $v_n = \sum_{k=0}^n p_n$, we may let $n \to \infty$ and obtain the completely different series

$$W(z) = \ln z - \ln\ln z - \sum_{k \ge 0} \ln(1 + p_k/v_k)\,, \tag{83}$$

which converges if $|W(z)| > 1$.

4.8 A continuous family of series and arbitrary-order iterative formulae

For any $v \ne 0$ we can put

$$p = \ln\left(\frac{z}{ve^v}\right) + 2\pi i k\,, \tag{84}$$

and we will have $W(z) = W(v\exp(v + p))$ so we may use (75). This gives us many interesting new series for W, but it also gives us a family of iterative formulas for W, which can be of arbitrary order, as follows. We choose $k = 0$ because we want $p \to 0$.

Choose v_0 to be an approximation to $W(z)$. Then for any v_n, define p_n by equation (84), and compute an improved estimate v_{n+1} by

$$v_{n+1} = v_n + \frac{v_n}{1 + v_n} p_n + \ldots + \frac{q_N(v_n)}{(1 + v_n)^{2N-1}} \frac{p_n^N}{N!}\,. \tag{85}$$

If $N = 1$, then the iteration is quadratically convergent (like Newton's Method). If $N = 2$, the iteration is cubically convergent. In general if $v_n \exp(v_n) = z(1 + \epsilon)$ then $p = \ln(1/(1 + \epsilon)) = O(\epsilon)$ and so the error in v_{n+1} is $O(\epsilon^{N+1})$, provided we are not too near the branch point $z = -1/e$.

We do not pursue here the issues of appropriate initial guesses v_0 and optimal efficiency by choice of N and the number of digits used.

4.9 Rearrangements

Some rearrangements include the series from [11] below, which use $L_\tau = \ln(1 - \tau)$ and $\eta = \sigma/(1 - \tau)$.

$$W_k(z) = \ln_k z - \ln\ln_k z - L_\tau + \sum_{n \ge 1} (-\eta)^n \sum_{m=1}^{n} (-1)^m \left[\begin{matrix} n \\ n - m + 1 \end{matrix} \right] \frac{L_\tau^m}{m!} \tag{86}$$

and

$$W_k(z) = \ln_k z - \ln\ln_k z - L_\tau + \sum_{m \ge 1} \frac{1}{m!} L_\tau^m \eta^m \sum_{p=0}^{m-1} \left\{ \begin{matrix} p + m - 1 \\ p \end{matrix} \right\}_{\ge 2} \frac{(-1)^{p+m-1}}{(1 + \eta)^{p+m}}\,. \tag{87}$$

The series converge for large enough real z, though the detailed regions of convergence are not yet settled. Curiously enough (87) is exact at $z = e$ and at $z = \infty$, and moreover if we truncate it to N terms it agrees with the N term Taylor series expansion at $z = e$ as well, making one think of 'Hermite' interpolation at e and at ∞. Convergence is rapid.

4.10 Some Other Asymptotic Series

The following are special cases of Theorem 2 in [21].

$$\ln W(z) = \ln\ln z + \sum_{n \ge 0} \frac{B_n(\ln\ln z)}{\ln^n z} \tag{88}$$

where the polynomials B_n are computable by $B_0 = 0$, $B'_{n+1}(x) = nB_n(x) - B'_n(x)$ and $B_n(0) = 0$.

Secondly, for any b and c,

$$e^{bW(z)} W(z)^c = z^b \ln^{c-b} z \sum_{n \ge 0} \frac{C_n(\ln\ln z)}{\ln^n z} \tag{89}$$

where the polynomials C_n are computable by $C_0 = 1$, $C'_{n+1}(x) = (b - c + n)C_n(x) - C'_n(x)$ and $C_n(0) = 0$ for $n > 0$.

More generally, for any b and c and any power series

$$G(x) = \sum_{i\geq 0} a_i x^i \tag{90}$$

with $a_0 \neq 0$, the following expansion holds:

$$e^{bW(z)} W(z)^c G\left(\frac{1}{W(z)}\right) = z^b \ln^{c-b} z \sum_{n\geq 0} \frac{A_n(\ln\ln z)}{\ln^n z} \tag{91}$$

where the polynomials A_n are computable by $A_0 = a_0$, $A'_{n+1}(x) = (b-c+n)A_n(x) - A'_n(x)$ and $A_n(0) = a_n$.

4.11 Series for $W(\rho\exp(it))$

If $y = W(\rho\exp(it))$ for $\rho \geq 0$ and $-\pi < t \leq \pi$, then define v by

$$W(\rho e^{it}) = W(\rho) + it + v\,. \tag{92}$$

Then $W\exp W = \rho\exp(it)$ implies

$$(W(\rho) + it + v)e^{W(\rho)+it+v} = \rho e^{it} \tag{93}$$

or, using $W(\rho)\exp(W(\rho)) = \rho$,

$$1 + \frac{it}{W(\rho)} + \frac{1}{W(\rho)}v = e^{-v}\,. \tag{94}$$

But this is just (70) with $\sigma = 1/W(\rho)$ and $\tau = -it/W(\rho)$, and thus all of our fundamental series solutions to (70) apply! The nicest one is (75), which splits into separate series for the real and imaginary parts of $W(\rho\exp(it))$. We have

$$W(\rho e^{it}) = \sum_{n\geq 0} \frac{q_n(W(\rho))}{(W(\rho)+1)^{2n-1}} \frac{(it)^n}{n!} \tag{95}$$

and clearly all the odd terms are purely imaginary and the even terms are real.

5 Infinite Products

From the relation $W(z) = z\exp(-W(z))$ it is easy to see that any series for $W(z)$ may be trivially transformed into an infinite product. For example, from the series (6) we have

$$W(z) = z\prod_{n\geq 1} \exp(\frac{(-n)^{n-1}}{n!} z^n) \tag{96}$$

but this of course gives us no essentially new information. However, the series (83) gives us

$$W(z) = \ln(z)\prod_{n=0}^{\infty} (1 + p_n/v_n) \tag{97}$$

in terms of the iterates of (76), and this is a simpler and more natural representation if nothing else.

6 A Final Pair of Expansions

The iterations (76–77) may be used to show that $W(z)$ can be written as

$$W(z) = \cfrac{z}{\exp\cfrac{z}{\exp\cfrac{z}{\exp\cfrac{z}{\ddots}}}} \tag{98}$$

or

$$W(z) = \ln\cfrac{z}{\ln\cfrac{z}{\ln\cfrac{z}{\ddots}}} \tag{99}$$

according as $|W(z)| < 1$ or $|W(z)| > 1$. These curious formulae are just the iterated exponential in disguise, and indeed are naturally discovered from rewriting $W(z) = z/\exp W(z)$ and $W(z) = \ln(z/W(z))$ as iterations.

Acknowledgements

We thank Dave Hare and Bruno Salvy for useful comments on earlier drafts. We also thank Gaston H. Gonnet for putting W into Maple in the first place.

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