Goldbach's Conjecture and Centers of a Network, Denise Vella-Chemla, April 2026.

Abstract : The conjecture called *even Goldbach's conjecture*, that has not recieved a proof until now, asserts that every even number n greater than 2 is the sum of two prime numbers. Is provided below a justification of this conjecture that is based on the idea of considering certain sums equal to n as points of $\mathbb{N}^2$ having pairs of coordinates of the form $(p, n-p)$, with p a prime number between 3 and $n-3$ included, then to considerate those points as vertices of a graph, and to see Goldbach's decompositions as centers of a punctured triangular graph based on that graph, the existence of such centers being ensured.

1. Introduction

The conjecture known as *Goldbach's even conjecture*, which dates from 1742 [12] in a letter from Christian Goldbach to Leonhard Euler, states that every even number n greater than 2 is the sum of two prime numbers. [1] [2] (see R.-C. Vaughan [26] for a historical account of his). This conjecture has interested many mathematicians (such as G. Cantor [8] who verified, at the time, by hand, that it was true for all numbers up to 1000, or C.-A. Laisant [17] who proposed an experimental procedure allowing it to be verified). A justification of Goldbach's conjecture is presented below, which was first based on the idea of considering certain sums equal to n as points in the plane $\mathbb{N}^2$. The numbers p_k and $n - p_k$, with p_k a prime number between 3 and $n-3$ inclusive, are placed, p_k, on the x-axis and $n - p_k$ on the y-axis.

2. Representation of the decompositions of n into sums $p_k + (n - p_k)$, with p_k a prime number, in the $\mathbb{N}^2$ plane

Let E_n be the set of weakened prime numbers between 3 and $n-3$ inclusive. We denote by E'_n the set of numbers $n - p_k$ with $p_k \in E_n$.

$$\begin{aligned} E_n &= \{p_k \mid 3 \leq p_k \leq n-3, p_k \text{ is a prime number}\} \\ E'_n &= \{n - p_k \mid p_k \in E_n\}. \end{aligned}$$

For convenience, we restrict the case study of an even number n to a square portion of $\mathbb{N}^2$: we represent the points useful for the justification by points (x, y) in a square C of side length n, a square whose vertices are the points $(0,0)$, $(n,0)$, $(0,n)$ and (n,n). The prime numbers p_k are positioned on the x-axis. The numbers $n - p_k$ are positioned on the y-axis [3].

We draw a network of lines in square C :

- Each vertical line has the equation $x = p_k$ for each prime number $p_k \in E_n$;
- Each horizontal line has the equation $y = n - p_k$ for each prime number $n - p_k \in E'_n$.

The set R of intersection points of the network of lines in the square C is defined by

$$R = \{(x, y) \in \mathbb{N}^2 \mid x \in E_n,\ y \in E'_n\}.$$

1. The Latin phrase "*Sed et omnis numerus par fit ex uno vel duobus vel tribus primis*," which can be translated as "*But every even number is composed of one, two, or three prime numbers*," is also found in a posthumous work published in 1701 by Descartes (cf C. Adam and P. Tannery [1], or G. Belgioioso [4].
2. We will consider here only even numbers greater than 4, as sums of two odd prime numbers.
3. Refer to a geometry book such as that of Michèle Audin [3].

Example : Let's take as examples the sets associated with even numbers $n = 16$ and $n = 24$: E_{16}, E'_{16}, E_{24} et E'_{24}. On a

$$\begin{array}{ll} \mathrm{E}_{16} = \{3,5,7,11,13\} & \mathrm{E'}_{16} = \{13,11,9,5,3\} \\ \mathrm{E}_{24} = \{3,5,7,11,13,17,19\} & \mathrm{E'}_{24} = \{21,19,17,13,11,7,5\} \end{array}$$

3. Graphic illustration of an example

To fix the ideas, let's illustrate the case $n = 16$: the Goldbach components of the number $n = 16$, 3 and 13 on the one hand, and 5 and 11 on the other, have their associated point $(3, 3)$, $(5, 5)$, $(11, 11)$ and $(13, 13)$ colored in red on the ascending diagonal.

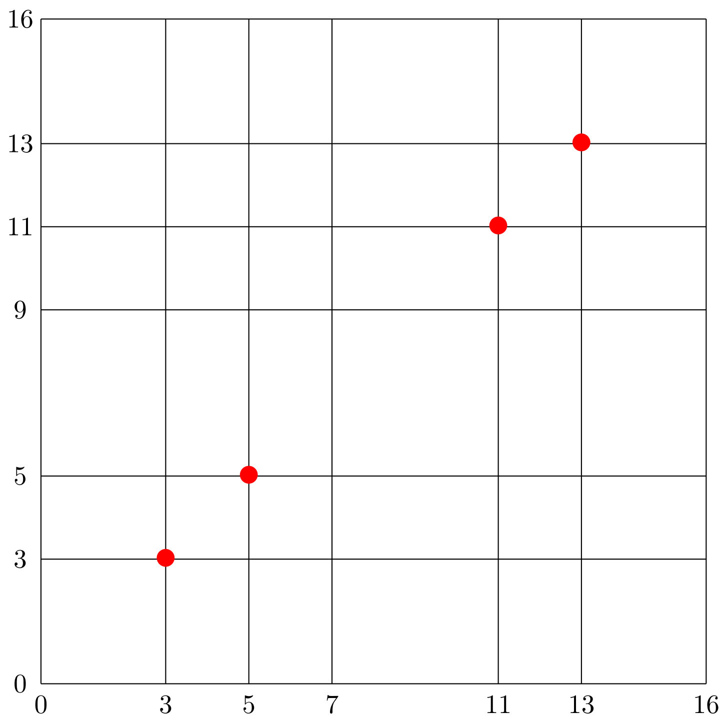

FIGURE 1 : The example of the square for $n = 16$. The Goldbach components of 16, belonging to $\{3, 5, 11, 13\}$, are colored in red on the ascending diagonal.

4. Graph, eccentricity, centers

Let us now adopt another perspective on the points of our network of points : let us see them as the vertices of a graph, linked by edges. In graph theory (see O. Cogis and C. Schwartz [9], J.-M. Meny, G. Aldon and L. Xavier [20], F. Buckley and F. Harary [6]), a graph is denoted $G = (S, A)$ with S a (here finite) set of vertices and A a (here also finite) set of edges between certain vertices. Graph theory originated in Euler's work on circuits crossing the Königsberg bridge once and only once. It was studied in France at the end of the 19th and beginning of the 20th centuries (see the works of C. Jordan [15], A. de Polignac [21], [22] or A. de Sainte-Laguë [24] [4])). Networks have become very familiar to us due to our massive use of the internet and navigation systems based on

4. This reference to the Sainte-Laguë's article was provided in this article of Alain Connes [10].

satellite positioning systems (GPS).

The labeled networks that model our problem are of a particular form ; every edge is the boundary of a chessboard square [5]. This implies that a path exists between any two vertices [6]. This property is expressed by saying that the graph is strongly connected.

The labels on the edges of the graph take the following values :

- the prime number 3 labels the edges of the first row and the first column ;
- the gaps between successive prime numbers label the edges of the following rows and columns except for the edges of the last row and those of the last column ;
- the edges of the last row and those of the last column have as their label the gap separating the largest prime number less than or equal to $n-3$ and n.

The labeling of edges is a function from A to $\mathbb{N}^+$ (see figure 2).

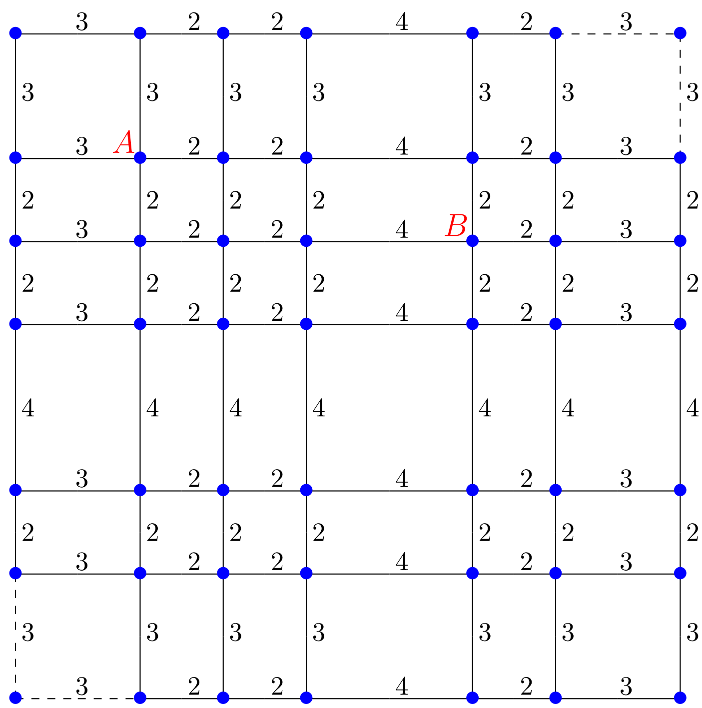

FIGURE 2 : Wedges labeling for $n = 16$.

The distance between two vertices s and s' of the graph is the minimum of the lengths of the paths leading from s to s'. The distance is a function from $\mathbb{N}^2$ to $\mathbb{N}^+$. The distance separating vertex A from vertex B, which we have noted in the graph, is equal to 10.

Let's now introduce the notion of eccentricity : the eccentricity of a vertex in a graph is the maximum of the distances separating it from each of the other vertices in the graph. This notion is important

5. see Laquière [18].
6. see A. Sainte-Laguë [24], bottom of page 15 and the notion of a complete circuit.

from an application point of view : take, for example, the positioning of an emergency care center : we want it to be positioned as centrally as possible.

$$e(s) = \max\{d(s, s')\}.$$

Since the set of vertices and the set of edges are two finite sets, the maximum exists and is well-defined on the set of paths from one vertex to each of the others.

To clarify the concepts, let's calculate the eccentricity of the vertices of the associated example graph $n = 16$ (the edges at the top right and bottom left of the diagram are marked with dotted lines, and this removal of 4 edges will be justified later) and note it on Figure 3.

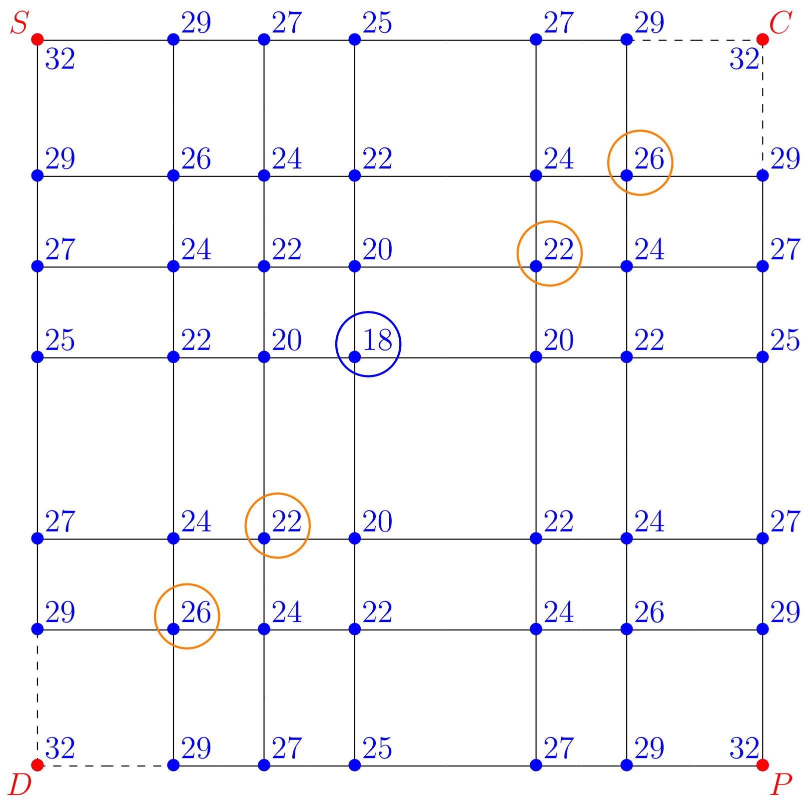

FIGURE 3 : vertices' eccentricities.

Mathematicians introduced the notion of the *center of a graph*. This notion was first introduced by C. Jordan in 1869 [15] for unlabeled graphs ; we find this notion in this article of A. de Sainte-Laguë [24], p. 9 : when there is no label on the edges, the distance between two vertices is simply the number of edges separating them. We adapt the notion of a *graph center* to the fact that the edges of the graphs used in our model are labeled by defining it as follows : a graph center will be a vertex whose eccentricity is minimal when the distance between two vertices is the sum of the labels of a path of minimum length separating them. Since the set of eccentricities of the different vertices of the graph is a finite set, the notion of minimum is well defined on this set of eccentricities of the different vertices of the graph.

We have circled the center in blue on the lattice. This point is, as expected, located in a fairly central position in the lattice.

Our goal being that the points corresponding to the Goldbach components of n are the centers of a graph, we decide to keep only the elements of the lattice below the descending diagonal of the square. The problem is that the bottom left corner could then be the only center ; this is why it is excluded from the new graph, for which we are going to find the centers : a punctured triangulated graph (we symbolized the exclusion of the "trivial centers" by the dotted edges on the graph in Figure 3).

Let's provide the fourth and final graph of the eccentricity calculation. and the visualization of the centers of the "punctured triangular graph" on figure 4.

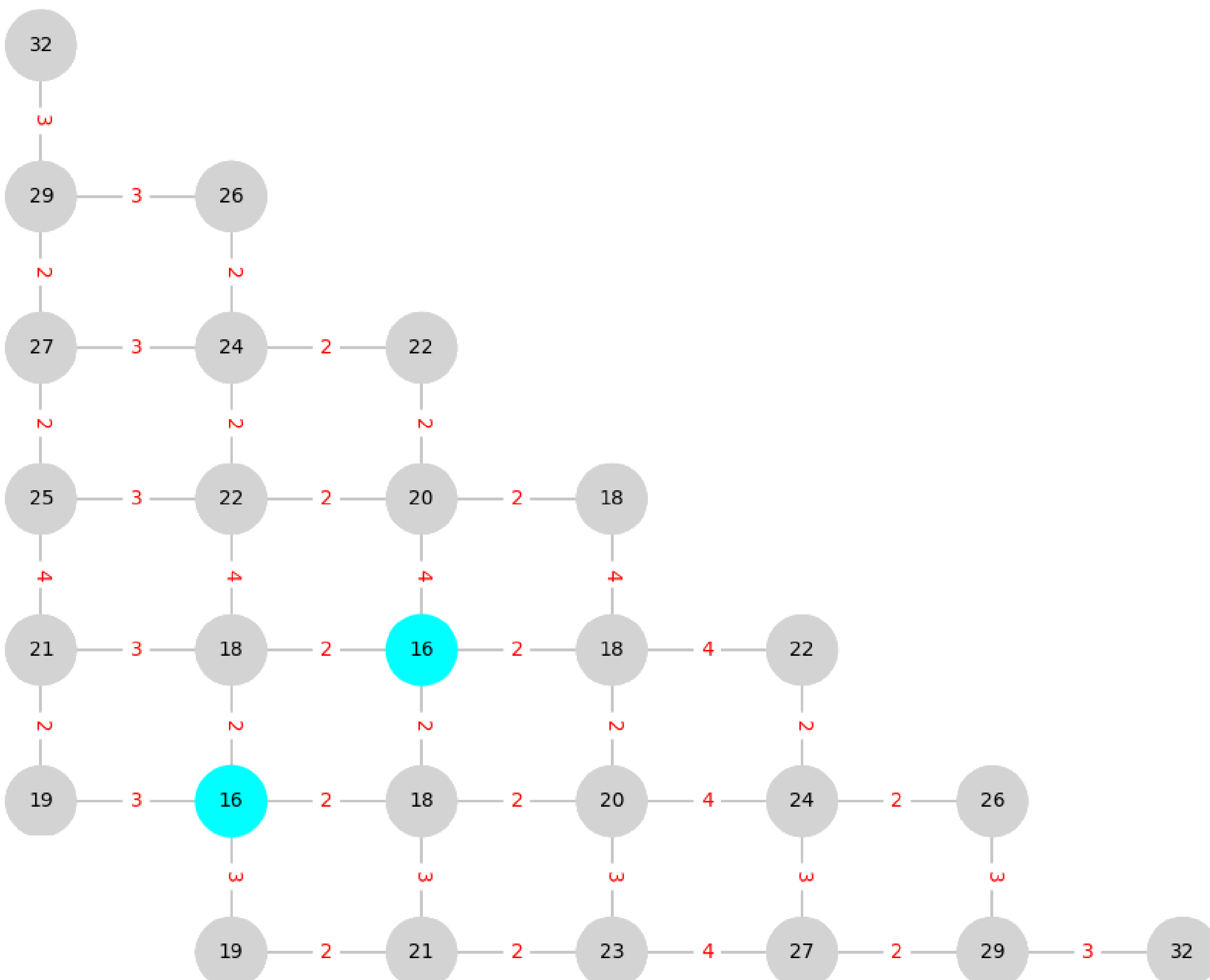

FIGURE 4 : Excentricités des sommets du nouveau graphe.

Each center of the graph associated with n corresponds to a Goldbach decomposition $p + q$ of the number n, where p and q are two prime numbers. One of the two prime numbers p or q is simply the sum of the labels of the vertical edges that lead to the center, while the other prime number is simply the sum of the labels of the horizontal edges. By construction, the sums in question are prime numbers : if we place the origin at the top left rather than at the bottom left (reasoning in the Cartesian plane), the successive numbers on the edges oriented from top to bottom or from left to right are labeled by the first odd prime number 3 and then by the gaps between successive prime numbers, the sum of such successive labels is necessarily a prime number. We understand that the centers are the minimum distance from the extreme points located at the top left and bottom right of the graph (any path to a point other than these two points can have its length increased by moving further towards these two points). Furthermore, when studying Goldbach's conjecture, we usually distinguish the even numbers that trivially satisfy the conjecture : these are the even numbers that are doubles of a prime number. In the model proposed here, Goldbach's trivial decomposition $2p = p + p$, where p is a prime number, corresponds to a center of the graph. Contrary to what one might have expected, such a center is not at the "Euclidean" center of the initial square : it is at the center in terms of the number of edges of the triangular graph point, that is to say, it is always located on the hypothenuse of the isosceles right triangle subtending the graph ; the path that leads to this center of the graph is also an isosceles right triangle.

The punctured isosceles right triangles visualizing the centers and therefore the Goldbach decompositions of the numbers n between 8 and 102 can be downloaded from this address. The program calculating those centers was provided by Gemini, following the author's instructions for labeling adequately edges, vertices, computing eccentricities, first in the squares and after in the punctured triangles (it can be downloaded at this address).

Readers wishing to deepen their knowledge concerning graph theory can study the references of G.Y. Handler [14], R. L. Francis and J. A. White [11], O. Kariv and S. L. Hakimi [16], F. Butelle [7], and J. A. Bondy and U. S. R. Murty [5] (page 109 of which deals with the notion of center of a graph) and S. Ratel [23] as well as the book by Jean-Pierre Serre [25].

5. Justification for the existence of a graph center

The existence of at least one center of the graph has been justified as the different definitions have been introduced. The strong connectivity of the graph, the fact that the sets (of vertices, edges, minimum path lengths, eccentricities) are finite sets guarantees the existence of the maxima and minima used in the calculations. The calculation of minimum-length paths falls within the theory of dioids (or semirings), also called min-plus algebras or tropical algebras (see M. Gondran and M. Minoux [13] as well as the work of Stéphane Gaubert and Marianne Akian [2]), and also within graph theory, which is very prominent in artificial intelligence and in the study of networks (communication, social, etc., see J.-L. Laurière [19]).

It is hoped that the justification provided above ensures the existence, for any even number $n > 4$, of two odd prime numbers of which it is the sum.

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Triangle 25x25 : Coin exclu (X, noir)
Centres du 'gros' graphe en CYAN