The Riemann flow and the zeros of zeta

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I will present you a space $(X, \underset{\mathbb{R}_+^*}{\circlearrowleft})$ (on classic real numbers) which will allow two things :

1. the first thing that it will give is a spectral interpretation of the zeros of the Riemann zeta function ;
2. and the second thing that it will do is that it will sort of reduce the Riemann hypothesis to justify (which I cannot do) a trace formula, and by trace formula I mean the following : I mean the simplest example of a trace formula is just the fact that if you take a matrix, the sum of the eigenvalues of the matrix (i.e. a_{ij}) is equal to the sum of the diagonal elements $\operatorname{Tr} M = \sum \lambda_j = \sum a_{ii}$.

The trace formula we have shown of course works with finite matrices valued by integers but what I want to explain is how, once we shall have the flow and so on, we shall get both the spectral interpretation and the fact that the Riemann hypothesis holds, provided the fact you can justify the trace formula.

Let me start with the very beginning, some of the relations already known in the Riemann's paper which is that if you start with the Riemann zeta function and you look at the number of zeros (I will call them $\rho_1, \rho_2, \ldots,$) you start plotting the function $N(E)$ which is the number of zeros which are non trivial and whose imaginary part is in the interval $]0, E]$ (i.e. $N(E) = \#$ of zeros such that $\mathfrak{Im}\, \rho \in]0, E]$). So you count how many zeros you have, accepted you know where they exactly are and you count these, and what you find is of course a staircase function, a step function as this, and this function turns out to be the superposition of two behaviours. So if I call this function $N(E)$, it is equal to the sum of two things, the sum of a very smooth and nice function which is the curve that I have plotted there, which can easily be computed, in fact it is given in Riemann's paper, it is equal to

$$N(E) = \langle N(E) \rangle + N_{\text{osc}}$$

with

$$\langle N(E) \rangle = \left(\frac{E}{2\pi}\right)\left(\log \frac{E}{2\pi} - 1\right) + \frac{7}{8} + o(1)$$

This first function is just given by the Stirling formula applied to the gamma factors *(mal audible)*. You have this term. And then you have another term, which is if you want the interesting part somehow, because instead of just being this smooth completely controlable function, it is the part which oscillates and which is of course necessary because we want to obtain a staircase function. So if you plot that oscillatory part, because it is just a difference that you have to calculate (for $N(E)$), the type of graph will be like this, you get a graph which oscillates so widely that if you take its integration over any interval then you get 0. So this oscillatory part has a formula for it, it is equal to :

$$N_{\text{osc}}(E) = \frac{1}{\pi} \operatorname{Im} \log \zeta\left(\frac{1}{2} + iE\right)$$

Vidéo visionnable ici https://www.youtube-nocookie.com/embed/yTpnamUP9Q4.
Transcription d'une vidéo mise en ligne le 16.10.2021, Denise Vella-Chemla, octobre 2021.

(the factor $\frac{1}{\pi}$ because you count an integer), and you have to take the branch of the log which comes from $+\infty$. And this function is in $O(\log E)$. And one very important observation which is motivating the problem of spectral interpretation of zeros is the following, it is due both theoretically to Hugh Montgomery and to an experimentalist Andrew Odlyzko ; the observation is the following : it's that if you separate the two behaviours, between the behaviour which is the average and the behaviour which is the oscillatory part, the best way to separate them is to rescale the zeros, in that we call x_j the average value evaluated on ρ_j (noté $x_j = \langle N(\rho_j) \rangle$), this makes that now this curve there is replaced by a step by step log 1 *(mal audible)*. Somehow the x_j means is equal to one. So what you do now, you count the number of pairs (i, j) in an interval $[1, M]$ of integers such that $x_i - x_j$ belongs to some interval $[\alpha, \beta]$ and what Montgomery proved in a certain range and what Odlyzko checked with very high precision is that this function is equivalent to

$$\#\{i, j \in [1, M]; x_i - x_j \in [\alpha, \beta]\} \sim M \int_\alpha^\beta \left(1 - \left(\frac{\sin \pi u}{\pi u}\right)^2\right) du$$

So that's a fact, which is conjectured in general, and which is numerically tested.

And now the amazing fact somehow is that this is statistics *(showing the integral)*. It is very different from statistics from random numbers. If you have random numbers, you don't have this integral behaviour, you just have a Poisson distribution.

And this statistics is exactly identical to the statistics of eigenvalues of a random hermitian matrix. This was observed in a famous encounter between Hugh Montgomery and Freeman Dyson, in Princeton. Dyson proved that this was the statistics of eigenvalues of a random hermitian matrix.

Now we come up with the following idea : where are random matrices occuring. Why were they invented ? They were invented by physicists, precisely because when they want to compute energy levels for atomic spectra, they found that it was a mess and the only way to obtain something was to deal with random matrices but the matrices they used were not hermitian. The matrices that you get for atomic systems are real symmetrical matrices, it's different.

So there is then an obvious problem which is that if those zeros are sort of statistics reinterpreted as eigenvalues of a suitable operator, this operator and the corresponding Hilbert space will get a name because this idea of interpreting zeros as eigenvalues of operators is an old idea. It is an idea which was began by Pólya and also Hilbert. So I will just use the terminology and I will call a Pólya-Hilbert space $\mathcal{H}$ this hypothetical Hilbert space and I will call D the hypothetical operator. This operator should be such that its zeros are exactly the non-trivial zeros of zeta and then, what you would like to prove is that

$$\left(D - \frac{1}{2}\right)^* = -\left(D - \frac{1}{2}\right)$$

(D is skew-adjoint).

Now there is another very important follow-up to this idea which is the work of Atle Selberg in which what Selberg did, really, is to construct a system, exactly as the two others did, namely by quantizing a classical system. What was the classical system in Selberg space, it was the geodesic flow on a suitable riemannian manifold M, and some of this geodesic flow of course gives by quantization the Laplacian acting in the metric L^2. However, the work of Selberg did give formulas which were very close to the formulas of number theory, and it is the hoped fact by two reasons.

The first reason is that somehow, whereas in the Riemann zeta function, you have a number of eigenvalues that grow by E (like $E(\log E)$), in the case of Selberg, we have something that grows faster.

The second thing is that whenever you take a geodesic flow, or in fact whenever you take a flow on a space which has time-reversal symmetry (time-reversal symmetry is the following thing, it is that you have a symmetry of the phase space : in the case of a geodesic flow, this symmetry is just that (p, q) goes to $(-p, q)$ (i.e. $(p, q) \to (-p, q)$). But that will hold for any Hermitian covering Hamiltonian of the form

$$H = \frac{1}{2m}p^2 + V(q)$$

($V(q)$ is some potential). So each time you will have a system that doesn't change when you pass from (p, q) to $(-p, q)$, you will have some trouble. Because when you quantize the system, you do not get an Hermitian matrix, but what you get is a real symmetrical valued matrix. Now the real symmetrical valued matrices don't have the same statistics as what you want. But there is another way to see it very quickly, which is that when you have a system that have that type of symmetry, there is a resonance between two orbits, two closed orbits of the system, the orbit over (p, q) is transformed when you take $(-p, q)$ instead of (p, q), so they will have the same length and so on and so forth.

Well, people are hunting for such a system and let me show you why this hunt, as it was set up, has food up : let me speak about the work of M. Berry and J. Keating, these are general ideas coming from quantum chaos, but what one should do then is to look for a classical system, typically a system like the flow over the manifold which under quantization will provided the good Pólya-Hilbert space operator. We can do the same type of computation that the one I was saying for the number of zeros of zeta, we can do it for the number of eigenvalues of the Hamiltonian which belong to the interval $[0, E]$ (i.e. $N_H(E) = \#$ of eigenvalues of $H \in [0, E]$). So you can extend this and what you will get is that it will break into two pieces : there is one piece which is quantized as a volume in the phase space (this is the average value) and there is another part which is the oscillatory part :

$$N_H(E) = \langle N_H(E) \rangle + N_{\text{osc}}(E)$$

You can actually expand this oscillatory part by an asymptotic expansion and what you find out is that this asymptotic expansion is covered by (you have to assume that the system is suitably chaotic and so on) but you can write an asymptotic expansion of the oscillatory part and it looks like this, for a system that have *(mal audible)* of dimension 2 (it is not an exact formula but it is an asymptotic formula) :

$$N_{\text{osc}}(E) \simeq \frac{1}{\pi} \sum_{\gamma} \sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{2 \,\text{sh}\left(\frac{m\lambda_\gamma}{2}\right)} \sin(mE\, T_\gamma^{\#})$$

γ labels the periodic orbits. But I label the periodic orbits only once, I don't label the traversal of the periodic orbits. And then (m index), you sum on the number of times you traverse the periodic orbits, in positive direction, and T_γ is the length of the *(mal audible)* ; λ_γ is called the instability exponent of the orbit. It is not an exact formula, it's something that you can use in practice and so on.

Now what you do if you want to have some information about this hypothetical flow, you should certainly be able to compare the formula that gives you this with the formula that you would get from this expression upstairs by replacing zeta by the Euler product formula. If you write log as a log of zeta, it's a log of a product, it's a sum, convergent, and each of the terms that appears in zeta is just $\dfrac{1}{1-\dfrac{1}{p^{\frac{1}{2}+iE}}}$, and you get, $-\log\left(1-\dfrac{1}{p^{\frac{1}{2}+iE}}\right)$

What will you get when you will expand this ? You will get another formula. Let me write the

$$N_{\text{osc}}^{\zeta}(E) \approx \frac{1}{\pi} \sum_{p=2,3,5,7,\ldots} \sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{p^{m/2}} \sin(m\,E \log p)$$

(writing the term $\dfrac{1}{p^{m/2}}$*, he says it looks right ; and when he writes the* sin*, he says "because it is an imaginary part so you would find a sine also").* Except that I made a mistake, so if you are bored, you can try to find the mistake, and you will see that this mistake is absolutely essential.

So let's ignore the mistake for the time, and let's write that this is positive. So I mean what you can say :

- Ⓐ the first thing that you can say is that for the flow we are looking for, the periodic orbits should be labeled by primes ; the lengths of the orbits, the periods, should be $\log p$, that's clear, and the stability exponent should be also equal to $\log p$ (i.e. $\exp \lambda_p = \log\, p$) ;

- Ⓑ The second thing we can say of course is also that there must be no time-reversal symmetry ; if you had reversal symmetry, it would mean that each orbit in fact in this count is actually counted twice and that's impossible, I can't have a 2 for an orbit ;

What is the mistake which I did ? I forgot a minus sign, an overall minus sign. You see if you are careful, when you compute the sign which is here, you'll find unfortunately there is an overall minus sign here *(putting a bold minus sign in front of the expression for* $N_{\text{osc}}^{\zeta}(E)$*).* Now that's really bad, this overall minus sign. Because you could try to say the things by putting major exponents but it will never be consistent with the sign minus one. Because it would be some i to some n, and it will never consist to give you a minus one. So that's a big problem.

(une remarque d'un auditeur : "Why there is a problem ?") Well you have two oscillations, there is one which sort of oscillate in a positive way and the other in a negative way. There is no way you can say you know.

So this is really like the very starting point, the reason why a relatively naive approach like the one of Dyson doesn't work. And if we wonder to understand what happens, what you need is broadely to extend the framework.

There is another thing of course that should be done with respect to this problem and it is to understand somehow what is the correct framework in which to think about zeta functions. Let me go to that and this framework is :

Global fields and Algebraic geometry.

The idea there is the following : after a little while, you find out that the zeta function which is associated to a field, which is the field of rational numbers $\mathbb{Q} = K$, and you find out that the only properties of this field which are really being used mainly *(mal audible)* that what you will have is that of course there will be zeta functions for fields like $\mathbb{Q}(\sqrt{2})$, the fields of algebraic numbers (*that he writes* $a\mathbb{Q}\#$). But this will be isomorphic to K. But it turns out that there is another special case of this fact "which are the rights fields ?" to handle the problem and these are fields of functions, these are functions fields over a curve Σ but now instead of working on finite fields of numbers, you work on finite fields of functions on Σ, $\mathbb{F}_q$. So what is more important is not that you have these two special cases really, what is mainly important is that the fields for which the problem is well posed and hopefully the answer is positive can be defined conceptually in a fruitful manner, and this is really what matters. These fields are technically called global fields but let me tell you their definition.

The definition is the following : a field K (I consider a field just countable, it has no topological structure). So K is global if and only if K sits inside a locally compact ring A which we have to assume to be non discrete and semi-simple, and $K \subseteq A$ and it's a subset in some cases equal. By this equal, I mean it is discrete and co-compact.

So what you find out from this equality is that even though the field itself has not a known topology, it is tied up by this relation to a locally compact ring which of course has a very interesting topology and somehow, it turns out moreover that this ring is unique. Namely K determines A and A determines K. And the Riemann hypothesis should be true for that situation, for all global fields. Now it turns out that you can specialize yourself in the function fields case, and what happens there is that you have something much better which is acting for you in the sense that you have some sort of a dictionary, some sort of bilingual text which allows you to translate immediately the properties that you find technically for the theory of corresponding zeta function, to translate them geometrically ; it's like you want as if the first thing that you get is that it turns out that the zeros of ζ have there a clear spectral interpretation :

The dictionary is as follows :

zeros of ζ	eigenvalues of the Frobenius acting on $\mathrm{H}^1_{\mathrm{et}}(\bar{\Sigma}, \mathbb{Q}_l)$
Functional expansion	Poincaré duality
Explicit formulas	Lefschetz formula
HR	Castelnuovo inequality

(First line of the dictionary) : The finite field $\mathbb{F}_q$ is not algebraically closed and you need to pass there by product acting on some strange space which is called the etale cohomology of the curve with values unfortunately not in complex numbers that would be too nice, but in l-adic numbers, where l is prime to the characteristic of the field. So somehow you don't really get the spectral interpretation but you almost get it because you do get a space which is not a complex Hilbert space but which is not so far.

Then the second thing that you have is that the functional equation is just Poincaré duality for this cohomology.

The next line which is extremely important is the explicit formulas. So what is an explicit formula for instance for SO(1), and if you understand that, you will understand that you know them : in SO(1), an explicit formula, it's a formula that relates zeros to the primes in the Euler product expansion. In SO(1), we saw how to compute the oscillatory term asymptotically given by this formula. It's typical of what you have in explicit formulas. So the explicit formula is in fact the same as a Lefschetz formula for the Frobenius. Namely what is this formula ? What does Lefschetz formula look like ? It is a Fourier that is just telling you that if you count the number of fixed points of some transformation φ, you can count them by the following :

$$\#\text{fix }\varphi = \text{Trace }\varphi/H^0 - \text{Trace }\varphi/H^1 + \text{Trace }\varphi/H^2$$

(Traces of action of varphi over H^0 (resp. H^1, resp. H^2)).

That's really what you have. All right. And then you can go ahead and you have a beautiful dictionary for understanding this bilingual text that allows you to think about things, because you have geometric notions. That was the thing I wanted to provide because generally I want to provide tools that allow to think geometrically.

Now we immediately see from the Lefschetz formula what was the trouble with this strange approach and why we had this problem of the overall minus sign.

From this picture, we immediately see the solution : when we look at this last formula *(showing H^1 in the first line of the dictionary)*, we see that the spectral realization is in H^1 ; but when we look in this last formula, we see that H^1 is preceded by a minus sign. H^0 is a very trivial space, it's a 1-dimension space and by duality, H^2 is also a trivial space in 1-dimension. In the formula, you see the overall minus sign before the part concerning H^1. So you are stuck, this tells you that you were wrong to look for spectral interpretation in the sense at what physicists called an emission spectrum. *(une remarque d'un auditeur, mal audible)*, il a raison mais AC dit qu'il reviendra à la remarque ultérieurement). Now the point is really the following : what is the lesson from that ? The lesson of that is that now we understanded what the problem was. The information C is the following :

Ⓒ The Pólya-Hilbert spectral interpretation of the zeros should be an absorption spectrum.

What do I mean by that ? I mean in the central physics : when you look at a very distant star, what you see in the light is that a certain number of lines are missing, it's a number of dark lines. You don't see dark with a number of white lines, what you see are missing lines. In other words, what I'm saying is that you should attend from that space as appearing not as such, but from its negative. In more fancy operational mathematical language, it should appear as a cokernel, it should appear as a difference. So that's extremely important of course. And there is another thing that we learn, by continuing with this dictionary, which is if you want what the group should be. If you look at the situation of a positive characteristic, namely the situation where we take a curve on our finite field, then what happens is that it's not a flow that you get, it's really a single transformation which is this Frobenius transformation. So what you have here is the naturals $\mathbb{Z}$. It's not a flow. That might worry you, because you might say "How am I going to find a general answer, if in one case I am looking for an action for $\mathbb{Z}$, and in the case faced with 0 characteristic, I am looking for an action of $\mathbb{R}$. Now what you very quickly find out is that because of class fields theory, these two groups in fact *(relating $\mathbb{Z}$ and $\mathbb{R}$ to a third object encompassing them)* have a general formulation. The true group, the group that you should look for that will act on the space is the following : it is $\mathrm{G} = \mathrm{GL}_1(\mathbb{A})/\mathrm{GL}_1(K)$. This group has a name, it is called the idele class group. An essential step in class field theory is that this group is up to a compact group isomorphic to $\mathbb{Z}$ when you take the situation of finite fields and it's isomorphic to $\mathbb{R}$ when you take the case of number fields. So somehow what you gather from this is that which will replace the Frobenius. And when we replace the group in general, it's not $\mathbb{Z}$ or $\mathbb{R}$ (the real line) but it is this group. *(So he writes the condition D.)*

D The group which acts is not $\mathbb{Z}$ or $\mathbb{R}$ but is the group $G = \mathrm{GL}_1(\mathbb{A})/\mathrm{GL}_1(K)$ (the invertible elements in $\mathbb{A}$ evaluated by $\mathrm{GL}_1(K)$.

This group has a name, it's called the idele class group, and the term idele precises the term adele. The term idele comes from an ideal in a number field and then ideal classes, and then ideles, and eventually adeles. So what was known from many theoricians is that this was the right framework for our topic.

This is the information that we have, but now of course, you would find the space. For another reason that I don't want to describe that is due to statistical mechanics and that is the first line of the dictionary of non commutative geometry, one is very immediately led to the following space and flow :

$$\text{Space } X = \mathbb{A}/\mathrm{GL}_1(K)$$

The space is extremely simple and that's the space I will consider. Namely I consider two adeles a and b and I say (K^* invertible elements of K, that are not 0) :

$$a \sim b \iff \exists q \in k^* \quad \text{such that} \ \ a = bq$$

That's my space. And it is sort of obvious that you have an action on the idele class group, namely the quotient on that space. Why ? Because $\mathrm{GL}_1(\mathbb{A})$ acts on $\mathbb{A}$ simply by this : if I take $j \in \mathrm{GL}_1(\mathbb{A})$ and if I take $x \in \mathbb{A}$, I will associate to them jx, their product. The simplest action that you can take, you just take the multiplicative group on the additive group. And because I want it to act not by $\mathrm{GL}_1(\mathbb{A})$, but by the quotient by $\mathrm{GL}_1(K)$, I have to divide, so I mean this is so simple *(mal audible)*.

So let me write the first theorem. The first theorem is the spectral interpretation.

To state it, there have been completely irrelevant little topics and I put them here : it's about the fact that this group G is a little bigger than $\mathbb{R}$ or $\mathbb{Z}$, it has a compact piece, so let me just remind you of that :

$$\begin{array}{ll} G = K \times \mathbb{R}_+^* & \quad \text{number fields} \\ G = K \times \mathbb{Z} & \quad \text{function fields} \end{array}$$

Don't worry too much about this K piece *(underlining K in the two lines above)*. This K is important because when I will describe the Hilbert space, I will have to decompose it according to the character of K. So this will introduce some notations but don't worry. You could restrict yourself to the case of Hilbert space which is a field of type K.

I am now going to give you an interpretation of the zeros and I will not even have to define the zeta functions in general, or the L-functions. That's the test for a spectral interpretation, you shouldn't have to define the zeta functions, you shouldn't have to analytically continue, but you should be able to define the zeros, rightable, without doing anything.

Theorem 1. *Let E be the restriction map from $L^2_\delta(X) \rightarrow L^2_\delta(G)$ (to be continued below)*

I will explain why I have to put the little decoration δ for L^2. What am I saying here is extremely simple : inside X, I have the group G because if I take an element of $\mathrm{GL}_1(G)$, if it is an invertible element of $\mathbb{A}$, it is an element of $\mathbb{A}$, OK ?! So inside my space X, I have a much more trivial space, which is the space G. So I consider the restriction map, I will define the space morphism later, and I call it E. What is $L^2_\delta(G)$, it is the regular representation of G, but I put a weight, which is a Sobolev weight, just to control the additivité de Hecke (grösse additivity), it's computely trivial. Now the claim is that then the space $\mathcal{H}$ which is the cokernel of E is the Pólya-Hilbert space.

But to explain to you how it is the Hilbert space, I have to explain to you what is the operator. But what do we have ? What we have is a representation of the group G.

Theorem 1. *Let E be the restriction map from $L^2_\delta(X) \rightarrow L^2_\delta(G)$ then the space $\mathcal{H}$, which is the cokernel of E is the Pólya-Hilbert space i.e. with G acting on X by W (and on G itself) you can decompose $\mathcal{H}$ as a direct sum*

$$\mathcal{H} = \bigoplus_{\chi \in \widehat{K}} \mathcal{H}_\chi,$$

let

$$D_\chi = \lim_{\epsilon \to 0} \frac{W(e^\epsilon) - W(1)}{\epsilon}.$$

χ is a character of K (i.e. is in $\widehat{K}$), I restrict the action of the group G to the compact subgroup K. I decompose of course the action according to characters of this compact subgroup. So I get a sum of Hilbert spaces. On each of these Hilbert spaces, let us develop the operator. U is the representation in $L^2_\delta(X)$, V is the representation in $L^2_\delta(G)$, and W is the representation in the quotient, G acting by W. It's clear because every space is sort of covariant constructive so it has a *(mal audible)* representation.

(réponse à une question) : The idele class group has a maximal compact subgroup that I call K, that's all..

Now let me write a statement that is much more precise :

(continuation of Theorem 1)
Then the spectrum of D_χ is discrete, purely imaginary, (it's not self-adjoint because it's like a generator of a group of isometry), and moreover, an element ρ belongs to the spectrum of G_χ if and only if two things hold (χ is called a grössencharakter (i.e. a Hecke character) :

$$\rho \in \mathrm{Sp}\ D_\chi \iff \begin{cases} (\chi, \frac{1}{2} + \rho) = 0 \\ \mathfrak{Re}\, \rho = 0 \end{cases}$$

I am not getting all zeros. I am only getting those zeros which have the correct real part. Moreover you can prove a priori (that's not difficult to see)... *(interruption : "You might get all zeros (mal audible)."* "Yeah, but you cannot prove it at this stage. This is very important, you see. You see, the whole strategy is to prove it later, but at this stage, what you have is the following situation.

You couldn't prove, it would be totally foolish, to hope that at this stage, you could get all zeros. That is foolish. Why is it foolish ? Because somehow if you want, it's true that an operator D_χ of that kind has a purely imaginary spectrum, this is like the spectra radius formula, when you apply the Frobenius, gradually, you shrink the spectum, it's a bit delicate. However, it would be foolish to expect that, the point is that somehow, if you want, the understanding of which zeros you get should follow, like in Selberg's case, from the trace formula. Somehow this is like a small epsilon story *(mal audible)*. However, let me tell the following fact : even for Riemann zeta, I didn't have to bring the analytic continuation, let alone to define the Riemann zeta function or the L-function, I didn't define the *(Lefschetz ? mal audible)*. What I am telling you is that just by a very simple story, of the additive group and the multiplicative group, of a global field, which I have written upstairs, out of the comparison between the additive and the multiplicative, fall down *(mal audible)* the zeros of the L-functions. You didn't have to define anything, somehow the problem presents itself as the problem of understanding the line. But it is not the real line, it is the adelic line. What I am saying is that if you try to understand the adelic line, you will immediately need a space like this to understand the zeros.

Now as I said, the big problem, now, the big issue is that ok, you have the zeros that are on the correct critical line but how do you know you have all of them. To do that, the natural thing to do... the first of all I should say is why it is L^2 and why it is not quantization. It is very clear. Why it is L^2 is because the space is as the space of Lagrangian leaves in a foliation when you *(mal audible)* from eigenspace.

Now let me go and the natural thing to do now is to compute the trace, namely what you want is to have a trace formula that computes the following trace of an integral of some function $h(g)$:

$$\text{Trace} \int_{g\in G} h(g)W(g)\ d^*g$$

where d^*g is the Haar measure of the group. G is a locally compact group, it has a Haar measure and so on, and now, what we would like to do is to compute this trace. Of course, from the spectral side, we know what is this trace. It is just equal to :

$$\text{Trace} \int_{g\in G} h(g)W(g)\ d^*g = \sum_{\rho} \widehat{h}(\chi,\rho)$$

(ρ, index of the sum, satisfies the two conditions appearing in the theorem 1 above). $\widehat{h}(k)$ is the Fourier transform of h evalluated at a Hecke character (grössencharakter) χ and ρ.

This is exactly left in Selberg's situation, you have on one side the spectral side, the computation of the trace, but now what you want is another trace. Here you have that the trace of a matrix is the sum of its eigenvalues. Now what you want is another formula that tells you that the trace of the matrix is given by the sum of its diagonal elements. And this is exactly where the Lefschetz formula enters. What do we have ? We have a space, which is $L^2_\delta(X)$ and we define h to be the cokernel. So what does it mean to be the cokernel ? It means that you will have like an exact sequence :

$$0 \to L^2_\delta(X) \to L^2_\delta(G) \to h \to 0$$

What is this telling ? It tells that if you want to compute the trace here *(underlining h)*, you should compute and have that the sum of the trace $L^2_\delta(X)$ and the trace h is equal to the trace $L^2_\delta(G)$. This is exactly where is the minus sign of course. Because out of such a formula, you are going to get that the trace on h is minus the trace on $L^2_\delta(X)$ plus the trace on $L^2_\delta(G)$. But the trace here *(underlining $L^2_\delta(G)$)* is very trivial to compute because it is the trace of the regular transformation, that's proportional to h of 1. So there is no problem. So you see that the whole issue is to understand what is the trace in this space *(surrounding $L^2_\delta(X)$)*.

What is the trace in this space ? What do we have ? We have a space X on which G is acting by transformation. We want to compute the trace of a group of diffeomorphisms for a flow acting on this space X. And what we have for that is exactly the group from Atiyah and Bott for a Lefschetz formula for instance, in the case of manifolds, and in more elaborated cases.

Let me make some recall : if you take a manifold M and φ a diffeomorphism on the manifold (i.e. $M, \varphi\circlearrowleft$). Then you can look at the following operator which is

$$(U\chi)(x) = \xi(\varphi(x)).$$

We can rewrite this as an integral :

$$(U\xi)(x) = \xi(\varphi(x)) = \int k(x,y)\xi(y)dy$$

where $k(x,y)$ is just the coordinates, it's the delta function $\delta(y - \varphi(x))$. I have just rewritten the kernel.

What should be the trace ? It should be :

$$\text{“Trace”}(U) = \int k(x,x)\,dx = \int \delta(x - \varphi(x))dx$$

Of course, it will only exist on the fixed points, otherwise the zeta function vanishes. And near of fixed points, you can make a change of variable, and this will give you that this integral is equal to :

$$\int \delta(x - \varphi(x))dx \;=\; \sum_{\varphi(x)=x} \frac{1}{|1 - \varphi'(x)|}$$

The denominator of the fraction is a determinant.

So in other words, I have done nothing but rewriting the fact that the trace of a map matrix is the sum of the diagonal elements, but because these diagonal elements are continuous, I have to introduce this transversalic factor which is there *(showing the last fraction above)*.

That immediately of course extends to flows. You know how to deduce I presume the Lefschetz formula from this : you just write the same formula for action on a differential form and then, in the denominator, you get not the absolute value of the determinant but you get the determinant.

So now this formula extends to flows, and what does it give you ? You take a flow now, it's a little bit more complicated, because you will have to distinguish between periodic orbits of the flow and zeros of the flow. There will be two types of contributions but essentially the formula will be the same. So let me write it first in a fancy way and then in an elegant manner. So for a flow, if you take the trace of the integral of $h(t)U(t)dt$:

$$\text{Trace} \int h(t)U(t)dt$$

(underlying $U(t)$, this is $\xi(f_\sigma(x))$) *(mal audible)* Use a transformation of function that transforms this, *(and writing before that)* in $(U_t\xi)(x)$. It's equal to this : you have a flow F_t, F_t is the exponential of the vector field (i.e. $F_t = \exp t\nu$). So let me write first the formula in fancy terms. What you will have is a sum of periodic orbits first, there will be fixed points so you will have periodic orbits, and each of them will contribute by the sum *(writing)* but now it will be for m that belongs to $\mathbb{Z}$:

$$\text{Trace} \int h(t)U(t)dt = \sum_\gamma \sum_{m\in\mathbb{Z}} T_\gamma^* \frac{h(mT_\gamma^*)}{|1 - F_{T/mT_\gamma^*}|} + \sum_{\nu_l=0} \int \frac{h(u)}{|1-F_u^*|} du$$

(T_γ^* is the length of the primitive orbit). This is the first contribution and you have another contribution which is the sum on the zeros of the flow and each zero contributes by an integral, the denominator is the tangent map of the flow.

Of course I don't want to have described the form because I want to have this formula by its ingredients. What is the beautiful formula ? It is the following, it is that it is a sum in fact, over all periodic orbits, whether there is a point or a non trivial orbit, and for each of them, you write an integral, even for that *(pointing the first sum)*, you write an integral. It's an integral over the isotropic group for any point in the orbit. So it would be *(mal audible)*

$$\begin{aligned} \text{Trace} \int h(t)U(t)dt &= \sum_\gamma \sum_{m\in\mathbb{Z}} T_\gamma^* \frac{h(mT_\gamma^*)}{|1 - F_{T/mT_\gamma^*}|} + \sum_{\nu_l=0} \int \frac{h(u)}{|1-F_u^*|} du \\ &= \sum_{\text{periodic orbit}} \int_{I_x} \frac{h(u)}{|1-F_u/|} \end{aligned}$$

Well d^*u is the Haar measure on the isotropy, which is normalized until it has covolume 1. When you have this factor here (*underlining T_γ^* in the first line of the last formula above*), it is telling you that the true measure is the counting measure times this factor. And this has covolume 1.

So this formula is not very difficult to proof, it's just spectral, and what is also not difficult to justify is that, because I put absolute value at the denominator instead of taking the determinant, this formula continues to hold when you take a local field, or if you take adeles and so on.

So what I will now write is simply the same formula for the action of the group G on that space. And I just go on and write it down. When you compute the periodic orbits of this flow, and you find that they are exactly parametrized by ranks, in fact by what number theoricists call places, there are more than primes because there are also primes at infinity which are called places. So what you get is a sum, you just compute it, naively, and then you get the same types of terms like in the preceding formula, an integral, and you compute the isotropy that corresponds to each place and you find that it is what is called the local field at the place t, namely you take your field K, you complete it at the place, and you get a non trivial subgroup which is contained as a subgroup in the group G, this inclusion is well known in class fields theory, so you are on the right track, now what do you get, you get a term which is $h(u^{-1})$ (u inverse) over $|1-u|_v$ (i.e. module of 1 minus u in the local field) and finally you have the multiplicative Haar measure d^*u normalized on the local field so it has covolume 1. That's what you get from the trace, by this computation.

$$\sum_{v \text{ a place}} \int_{K_t^* \subset G} \frac{h(u^{-1})}{|1-u|_v} d_u^*$$

Now what you do is that you open a book of number theory, what are the ingredients of André Weil, the book in which he took the Riemann work and he tried to rewrite this very complicated explicit formula of Riemann (where are involved principal values and so on), and he find himself this formula *(showing the last formula he has framed)*. In other words, this formula is computed by Weil and what is it ? What is it equal to ? Well, it is equal to the following ? Weil tells that this is equal to the h Fourier transform at 0 plus the Fourier transform of h at 1 minus (and this is again our minus sign) the sum over of zeros of L-functions (the index is $L(\chi, \rho) = 0$, *(writing the ρ, he tells "and you take it at Hecke character (grössencharakter)", like here, showing the preceding Hecke character grössencharakter on a far blackboard)* of the Fourier transform of χ and ρ.

$$\sum_{v \text{ a place}} \int_{K_t^* \subset G} \frac{h(u^{-1})}{|1-u|_v} d_u^* = \widehat{h}(0) + \widehat{h}(1) - \sum_{L(\chi,\rho)=0} \widehat{h}(\chi, \rho)$$

Now you look back, and it's not difficult to see (I have not enough time to define the space $L^2_\delta(X)$ but it's very easy to see that when you define it, you have to impose that the function vanishes at zero, and that its Fourier transform vanishes at zero, because the point 0 and the space X are fixed by the action of the group G.

So what you find when you compute this trace... is that if you could, well that is very the mainly point. The mainly point is the following : If one can justify the trace formula there (which was just computed by Lefschetz), than one gets the following :

$$\widehat{h}(0) + \widehat{h}(1) - \sum_{\chi=0} \widehat{h}(\chi, \rho) = \widehat{h}(0) + \widehat{h}(1) - \sum_{\substack{\chi=0 \\ \mathfrak{Re}(\chi)=\frac{1}{2}}} \widehat{h}(\chi, \rho)$$

(on the left side, the sum is computed over all zeros, on the right side, only on those that are on the critical line).

Now it is very easy to deduce from that the lambda test function h *(mal audible)*.

Let me just say one thing. The Weil formula was very astonishing for the first time for me *(? ? mal audible)* and there is one fact which is completely obvious on this form. Let me tell you what it is. It's that each time the function h is positive, in the naive sense, that $h(u)$ is positive for every value of u, then this form *(the last written)*, is positive, that's completely obvious. So somehow what you know is that if you wanted to obtain a trace formula, that couldn't be a trace formula from a too complicated quantum theoretic nature, because it's not true in general that when you take the trace of ... operator, it will be positive. *(mal audible)*. It's only true for the termination of the system. This formula was graded for interpretation, exactly as for Frobenius, by permutation of the space.

For some time, I have really tried to justify this formula rigorously, there are problems and one of them is that you have principal values, exactly as in Selberg's, and so on. There is a very nice Selberg's fact that is that when you compute the Selberg's values, you find exactly the same as in André Weil's formula, and they involve the Euler constant and a $\log(2\pi)$, it is scalar *(mal audible)*. After a while, I sought to believe that instead of trying to justify on the nose this formula, mightly what I had to try to do was simply to say the following : one is exactly in the same situation as in the case of finite fields in which one has a curve, and the role of the curve... you see, the curve is like the difference between G and X. It is a circle and when you compute, you find exactly the right orbits of Frobenius. One has to replace for the Frobenius, one has a spectral interpretation, like the first time evolutional and one has the other lines of the dictionary for instance the Lefschetz formula for instance with explicit formulas. It is possible that Riemann hypothesis is much deeper.

These are perhaps only the first lines of the demonstration, but my belief, and this is why I wanted to talk in this meeting is the following : it's that I believe that it is by understanding better and better the dynamical aspects of this space and what is the geometry behind this space that we will be able to exactly imitate what André Weil has done in the case of curves and finite fields, and eventually, make it obvious, so it will fall, like a ripe fruit. So I take this as an extremely strong ambition from not being bounded, and limitate the study of geometric classical spaces, so to understand what is the geometry of space X in non-commutative framework.